Characterizations of Connected Perfect Domination in Graphs

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Abstract:

A subset S of $V\ (G)$ is a *connected perfect dominating set* of G if S is both connected and perfect dominating set of G. The *connected perfect domination number* of G denoted by $\gamma_{cp}(G)$ is the cardinality of the smallest connected perfect dominating set of G. A *connected perfect dominating set* of G with cardinality equal to $\gamma_{cp}(G)$ is called a $\gamma_{cp} - set$ of G. This paper shows some charaterization of a connected dominating set and the values or bounds of the parameter were determined. It also characterizes the connected perfect dominating set in the join and corona of graphs and the corresponding values of the parameter were also determined.

Keywords: *Domination, connected domination, perfect domination, connected perfect domination.*

INTRODUCTION

Let $G = (V\ (G), E(G))$ be a graph and $v \in V\ (G)$. The *open neighborhood* of v is the set $N_G(v) = N(v) = \{ u \in V\ (G): uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. If $S \subseteq V\ (G)$, then the *open neighborhood* of S is the set $N_G(S) = N(S) = \bigcup_{v\in S} N_G(v)$ and the *closed neighborhood* of S is the set $N_G[S] = N[S] = S \cup N(S)$.

A subset S of $V\ (G)$ is a *dominating set* of G if for every $v \in V\ (G)\backslash S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $NG[S] = V\ (G)$. It is a *connected dominating set* of G if the subgraph $\langle S\rangle$ induced by S is connected. It is a *perfect dominating set* of G if for each $v \in V\ (G)\backslash S$, is adjacent to exactly one vertex in S. The *perfect domination number* of G denoted by $\gamma_p(G)$is the cardinality of the smallest perfect dominating set of G. A perfect dominating set of G with cardinality equal to $\gamma_p(G)$ is called a $\gamma_p - set\ of\ G$.

Connected domination was investigated in [1] and [6] where the bounds and some properties of this type of dominating set were characterized. On the other hand, the perfect dominating set was investigated in [2], [4], [5]and [7]where the bounds and some properties of the perfect dominating set were characterized.

A subset S of $V\ (G)$ is a *connected perfect dominating set* of G if S is both connected and perfect dominating set of G. The *connected perfect domination number* of G denoted by $\gamma_{cp}(G)$ is the cardinality of the smallest connected perfect dominating set of G. A connected perfect dominating set of G with cardinality equal to $\gamma_{cp}(G)$ is called a $\gamma_{cp} - set\ of\ G$.

The next section presents some results of the concepts on connected perfect dominating set in a graph.

RESULTS

Remark 1. For a connected nontrivial graph G, $1 \leq \gamma_{cp}(G) \leq n-2$.

To see this, consider the graphs shown in Figure 1, K_2 and C_4. It can be verified that $\gamma_{cp}(K_2) = 1$ and $\gamma_{cp}(C_4) = 4-2 = 2$.

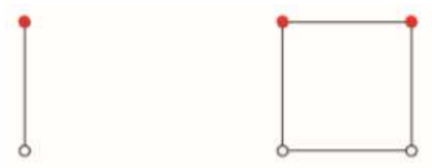

FIGURE 1. The graphs K_2 and C_4.

Theorem 2. Let G be a connected nontrivial graph. Then, $\gamma_{cp}(G) = 1$ if and only if $G = K_1 + H$ for some graph H.

Proof. Assume that $\gamma_{cp}(G) = 1$. Then the graph G has a connected perfect dominating set, say $S = \{a\}$ for some $a \in V(G)$. Since S is a perfect dominating set, $az \in E(G)$ for all $z \in V(G) \setminus S$. Take $H = \langle V(G) \setminus S \rangle$. Then $G = K_1 + H$. For the converse, suppose that $G = K_1 + H$ for some graph H. Take $S = V(K_1)$. Then S is a connected perfect dominating set of G. By Remark 4.1.1, $\gamma_{cp}(G) = |S| = 1$. ■

Corollary 3. For any complete graph K_n of order $n \geq 2$, $\gamma_{cp}(K_n) = 1$.

Proof. Let K_n be a complete graph of order $n \geq 2$. Then, $K_n = K_1 + \langle V(K_n) \setminus V(K_1) \rangle$. By Theorem 4.1.2, $\gamma_{cp}(K_n) = 1$. ■

Corollary 4. For any fan F_n where $n \geq 2$, $\gamma_{cp}(F_n) = 1$.

Proof. Let F_n be a fan of order $n+1$.Then $F_n = K_1 + P_n$. By Theorem 4.1.2, $\gamma_{cp}(F_n) = 1$. ■

Corollary 5. For any wheel W_n of order $n+1$, $\gamma_{cp}(W_n) = 1$.

Proof. Let W_n be a wheel of order $n+1$. By definition, $W_n = K_1 + C_n$. By Theorem 4.1.2, $\gamma_{cp}(W_n) = 1$. ■

Corollary 6. For any star $K_{1,n}$ of order $n+1$ where $n \geq 1$, $\gamma_{cp}(K_{1,n}) = 1$.

Proof. Let $n \geq 1$ be an integer. Then the star $K_{1,n} = K_1 + \overline{K_n}$. By Theorem 4.1.2, $\gamma_{cp}(K_{1,n}) = 1$. ■

Theorem 7. Let G and H be any connected nontrivial graphs. Then $S \subseteq V(G+H)$ is a connected perfect dominating set of $G+H$ if and only if S is a singleton dominating set of G or in H.

Proof. Let G and H be any graph. Suppose that S is a connected perfect dominating set of $G + H$. Since $G+H$ are connected graphs and S is a perfect dominating set, it follows that $S \subseteq V(G)$ or $S \subseteq V(H)$.Suppose $S \subseteq V(G)$ and $S \geq 2$.There is a contradiction since $|N(w) \cap S| \geq 2$

for all $w \in V(H)$. Hence, $|S| = 1$ i.e., S is a singleton dominating set of G. Similarly, if $S \subseteq V(H)$, then S is a singleton dominating set of H.

For the converse, suppose first that S is a singleton dominating set of G. Then S is a connected dominating set of $G + H$ by the definition of the join of two graphs. Since $|S| = 1$, and S is a dominating set, $|N(w)S| = 1$ for all $w \in V(G + H)\backslash S$. Hence, S is a connected perfect dominating set of $G + H$. Similarly, S is a connected perfect dominating set of $G + H$ whenever S is singleton dominating set of H. ■

Corollary 8. For any connected nontrivial graphs G and $H, \gamma_{cp}(G + H) = 1$ if and only if $\gamma(G) = 1$or $\gamma(H) = 1$.

Proof. Suppose $\gamma_{cp}(G + H) = 1$. Then $G + H$ has a connected perfect dominating set S say $S = \{a\}$ for some $a \in V(G + H)$. Suppose $S \subseteq V(G)$. By Theorem 4.1.7, S is a singleton dominating set of G. Hence $\gamma(G) = |S| = 1$. Similarly, if $S \subseteq V(H)$, S is a singleton dominating set of H by Theorem 4.1.7 and $\gamma(H) = 1$. For the converse, suppose first that $\gamma(G) = 1$. Then G has a singleton dominating set of G. By theorem 4.1.7, S is a connected perfect dominating set of $G + H$. By Remark 4.1.1, $\gamma_{cp}(G + H) = 1$. Similarly, if $\gamma(H) = 1$, it follows that $\gamma_{cp}(G + H) = 1$by Theorem 4.1.7 and Remark 4.1.1. ■

Theorem 9. Let H be a connected nontrivial graph. Then $S \subseteq V(K_1 + H)$ is a connected perfect dominating set of $K_1 + H$ if and only if $S = V(K_1)$ or S is a singleton dominating set of H.

Proof. Let H be a connected trivial graph. Suppose S is a connected perfect dominating set of $K_1 + H$. If $S = V(K_1)$, then we are done. Suppose $S \subseteq V(H)$. Then S is a dominating set of H. Let $v \in V(K_1 + H)\backslash S$. Since S is a perfect dominating set of $K_1 + H$, $|N(v) \cap S| = 1$. It follows further that $|S| = 1$. Hence, S is a singleton dominating set of H.

For the converse suppose first that $S = V(K_1)$. Then by the definition of the join of graphs S is a connected dominating set of $K_1 + H$. Let $z \in V(H)$. Then by the definition again of the join of graphs, $|N(z) \cap S| = 1$. Thus, S is a perfect dominating set of $K_1 + H$. Therefore, S is a connected perfect dominating set of $K_1 + H$. Now, suppose that S is a singleton dominating set of H. Then by the join of graphs, S is a connected dominating set of $K_1 + H$. Since $|S| = 1$, $|N(w) \cap S| = 1$for all $w \in V(K_1 + H)$. Thus, S is a perfect dominating set of $K_1 + H$. Therefore, S is a connected perfect dominating set of $K_1 + H$. ■

Corollary 10. Let H be any connected non-trivial graph. Then $\gamma_{cp}(K_1 + H) = 1$.

Proof. Let S be a $\gamma_{cp} - set$ of $K_1 + H$. By Theorem 4.1.9, S = $V(K_1)$ or S is a singleton dominating set of H. If $|S| = V(K_1)$, then $\gamma_{cp}(K_1 + H) = 1$. If S is a singleton dominating set of H, then $\gamma_{cp}(K_1 + H) = 1$. ■

Theorem 11. Let $m \geq 2$ and $n \geq 2$ be integers. Then $S \subseteq V(K_{m,n})$ is a connected perfect dominating set of $K_{m,n}$ iff $S = S_1 \cup S_2$ where S_1 and S_2 are singleton subsets of $V(\overline{K_m})$ and $V(\overline{K_n})$, respectively.

Proof. Let $m \geq 2$ and $n \geq 2$ be integers. Suppose that S is a connected perfect dominating set of $K_{m,n}$. Since $K_{m,n} = \overline{K_m} + \overline{K_n}$, $S = S_1 \cup S_2$ where $\phi = S_1 \subseteq V(\overline{K_m})$ and $\phi = S_2 \subseteq V(\overline{K_n})$. Suppose that $|S| \geq 2$. This contradicts the assumption that S is a perfect dominating set. Hence, $|S| = 1$ ie. S is a singleton subset of $\overline{K_m}$. Similarly, S_2 is a singleton subset of $\overline{K_n}$.

For the converse, suppose $S = S_1 \cup S_2$ where S_1 and S_2 are singleton subsets of $\overline{K_m}$ and $\overline{K_n}$ respectively. Then, S is a connected dominating set of $K_{m,n}$. Let $v \in V(K_{m,n})\backslash S$. Suppose that $v \in V(\overline{K_m})\backslash S_1$, then $|N(v) \cap S| = |N(v) \cap S_2| = 1$. Also, if $v \in V(\overline{K_m})\backslash S_2, |N(v) \cap S| = |N(v) \cap S_1| = 1$. Hence, S is a connected perfect dominating set of $K_{m,n}$. ■

Corollary 12. For any complete bipartite graph $K_{m,n}$, when $m \geq 2$ and $n \geq 2, \gamma_{cp}(K_{m,n}) = 2$.

Proof. Let $S = \{a, z\}$ when $a \in V(\overline{K_m})$ and $z \in V(\overline{K_n})$. Then $\{a\}$ and $\{z\}$ are singleton subsets of $V(\overline{K_m})$ and $V(\overline{K_n})$, respectively. By Theorem 4.1.11, S is a connected perfect dominating set of $K_{m,n}$. Since no singleton set is a connected perfect dominating set of $K_{m,n}$ it follows that $\gamma_{cp}(K_{m,n}) = |S| = 2$. ■

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