The boundedness of the solutions of degenerate divergent linear elliptic equations.

Kerimova M.

Institute of Mathematics and Mechanics
V.Bahabzade 9, AZ 1141, Baku, Azerbaijan

Abstract: *We prove the boundedness estimates of solutions for degenerate elliptic-parabolic equations.*

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1 Introduction

Many applied problems is reduced to degenerate elliptic equations. For example, drift-diffusion processes in porous media, differently chemistry problems and so on. In papers [1]-[10] some correspondingly results is considered.

We classes degenerate linear elliptic equations is considered. Let in $Q_T = \Omega \times (0,T)$, where $\Omega \subset R^n$, $n \geq 2$ bounded domain with smooth boundary $\partial\Omega$, problem

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) - \psi(x,t)\frac{\partial^2 u}{\partial t^2} +$$

$$+\sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u = 0, \tag{1.1}$$

$$u(x,t) = f(x,t), \quad (x,t) \in \Gamma = (0,T) \times \partial\Omega \tag{1.2}$$

$$u(x,0) = h(x), \quad x \in \Omega \tag{1.3}$$

is considered.

We consider the problem (1.1)-(1.3) under standard conditions for the functions $a_{ij}(x,t), b_i(x,t), c(x,t)$. Our main specific assumptions relayted with weight functions $\psi(x,t)$. About this conditions see [11].

A function $u(x,t) \in L_2(0,T; \overset{\circ}{W}{}^{1,2}_{2,\psi}(Q_T))$ is called of solution of problem (1.1)-(1.3) the integral identities

$$\int\limits_{Q_T} \frac{\partial u}{\partial t}\bar{\varphi}dxdt + \int\limits_{Q_T}\left[\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\frac{\partial \bar{\varphi}}{\partial x_i}\right) +\right.$$

$$\left.+\sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i}\bar{\varphi} + c(x,t)u\bar{\varphi}\right] dxdt+$$

$$+\int\limits_{Q_T} \psi(x,t)\frac{\partial^2 u}{\partial t^2}\bar{\varphi}dxdt = 0 \tag{1.4}$$

hold for any functions $\bar{\varphi} \in C^\infty(\bar{Q}_T)$ vanishing near Γ and almost everywhere $\tau \in (0,T)$

$$u(x,t) - f(x,t) \in L_2(0,T;\overset{\circ}{W}{}^{1,2}_{2,\psi}(Q_T)). \tag{1.5}$$

Theorem 1.1. *Let the standard conditions relayted the coefficients and weight functions for problem* (1.1)-(1.3) *be satisfied.Then there exists a constant M depending only on known parameters such that*

$$ess\sup\{u(x,t) : (x,t) \in Q_T\} \le M. \tag{1.6}$$

Proof. We shall prove estimates for u separately for the sets $\{u > 0\}$ and $\{u < 0\}$. We shall use the estimate

$$\int\limits_\Omega \left\{\frac{1}{2} + [u-m_0]_+^3\right\}^{r+1} dx+$$

$$+\int_0^r \int\limits_\Omega \omega^2(x)\Phi^{(r)}(u_k)\cdot\chi(m_0 < n < k)\cdot\left|\frac{\partial u}{\partial x}\right|^2 dxdt+$$

$$+\int_0^\tau \int\limits_\Omega \psi^2(x,t)\Phi^{(r)}(u_k)\chi(m_0 < n < k)\left|\frac{\partial^2 u}{\partial x^2}\right|^2 dxdt+$$

$$+\frac{r+1}{k(r)}\int_0^r \int\limits_\Omega (1+|u|)[u_k - m_0]\Phi^{(r)}(u_k)dxdt. \tag{1.7}$$

For get this estimate we substitute to integral identities (1.4) correspondingly test function, where

$$\Phi^{(r)}(u) = [u-m_0]_+\left[\frac{1}{2} + [u-m_0]^3\right]^r.$$

Let $\{\varphi_j(x)\}, j = 1,2,\ldots,J$, be a partition of unity such that

$$\sum_{j=1}^J \varphi_j(x) = 1, \left|\frac{\partial\varphi_j}{\partial x}\right| \le \frac{C_0}{R} \quad \text{for} \quad x \in \Omega,$$

$$\varphi_j(x) \in C^\infty(R^n), \text{supp}\varphi_j(x) \in B(x_j,R), J \le \frac{C_0}{R^n}, r < 1.$$

where $B(x_j, R)$ is a ball of radius R with centre $x_j \in \Omega$, C_0 is a number depending only on n. The number R will be choosen later on. We test the integral identity (1.4) with the function

$$\bar{\varphi} = \sum_{j=1}^{J} u_k \Phi^{(r)}(u_k)[u - u_j]\varphi_j(x), u_j = u(x_j, t). \tag{1.8}$$

Integration with respect to t yields

$$\int\limits_{Q_T} \omega(x)\Phi^{(r)}(u_k) \left|\frac{\partial u}{\partial x}\right|^2 dxdt = J_1 + J_2 + J_3$$

where

$$J_1 = -\sum_{j=1}^{J}\sum_{i=1}^{n}\int\limits_{Q_T} \omega(x)\Phi_1^{(r)}(u_k)|u - u_j|\varphi_j \frac{\partial u_k}{\partial x_i}\frac{\partial u}{\partial x_j}dxdt,$$

$$J_2 = -2\sum_{j=1}^{J}\sum_{i=1}^{n}\int\limits_{Q_T} \omega(x)\Phi_1^{(r)}(u_k)|u - u_j|\varphi_j \frac{\partial \varphi_j}{\partial x_j}\frac{\partial u}{\partial x_j}dxdt,$$

$$J_3 = -\sum_{j=1}^{J}\int\limits_{Q_T} |u - f|u_k\Phi_1^{(r)}(u_k)|u - u_j|\varphi_j dxdt.$$

For $\Phi_1^{(r)}(u_k)$ at $r \geq \frac{1}{2}$, $k > (m_0 + \frac{1}{2})$ have estimate

$$\Phi_1^{(r)}(u_k) \leq c_1(r+1)[u_k\Phi^{(r)}(u_k)\chi(m_0 < n < k)].$$

Later we get estimates for J_1 in case $\{u > 0\}$. We assume that $r \geq -\frac{1}{2}$ and choose the number R according to

$$R^{\varepsilon} = \frac{\varepsilon}{(r+1)^2}, \quad \varepsilon < \frac{1}{4}.$$

Then

$$|J_1| \leq \varepsilon \left\{ \int\limits_{Q_T} \omega(x)u_k\Phi^{(r)}(u_k) \left|\frac{\partial u}{\partial x}\right|^2 dxdt + \right.$$

$$\left. +C_2\frac{1}{(r+1)^2}\int\limits_{Q_T} \omega(x)u_k\Phi^{(r)}(u_k)\chi(m_0 < n < k)\left|\frac{\partial u}{\partial x}\right|^2 \right\} dxdt.$$

By Cauchy inequality we have

$$|J_2| < \int\limits_{Q_T} \omega(x)u_k\Phi^{(r)}(u_k)\left\{\varepsilon_1\left|\frac{\partial u}{\partial x}\right|^2 + \frac{1}{\varepsilon_1}\frac{C_3}{R^{n+2}}\right\} dxdt.$$

Correspondingly we have estimate J_3. Applying the last estimate and choosing ε small enough we get

$$ess \sup_{\tau\in(0,T)} \int\limits_{\Omega} \left\{\frac{1}{2} + [u_k(\tau,x) - m_0]_+^3\right\}^{r+1} dx+$$

$$+\int\limits_{Q_T} \omega^2(x)\Phi^{(r)}(u_k)\chi(m_0 < n < k)\left|\frac{\partial u}{\partial x}\right|^2 dxdt \leq$$

$$\leq C_4(r+1)^{\lambda_1}\left\{\int\limits_{Q_T} \Phi^{(r)}(u_k)[u_k - m_0]^2[u + |f|]dxdt + 1\right\}$$

with $\lambda_1 = 2(n+2)+2$.

We want to apply Moser iteration with respect to the integral

$$I_k(r) = \int\limits_{Q_T} \Phi^{(r)}(u_k)[u_k - m_0]^2[u + |f|]dxdt. \tag{1.9}$$

To this end we use the embedding inequality

$$\int\limits_0^T \left\{\int\limits_{\Omega} |v(x,t)|^{2(1+\frac{2p}{n})}dx\right\}^{\frac{1}{p}} dt \leq$$

$$\leq C_5\left\{ess \sup_{t\in(0,T)} \int\limits_{\Omega} v^2(x,t)dx\right\}^{\frac{1}{p}+\frac{2}{n}-1} \int\limits_{Q_T} \left|\frac{\partial v}{\partial x}\right|^2 dxdt \tag{1.10}$$

which is fulfilled for $1 \leq p < \frac{n}{n-2}$ and arbitrary function $v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^1_{2,\omega}(\Omega))$. Applying Holders inequality to (1.9) we obtain

$$I_k(r) \leq C_6\int\limits_0^T \left\{\int\limits_{\Omega} [\Phi^{(r)}(u_k)[u_k - m_0]^2]^p dx\right\}^{\frac{1}{p}} dt \leq$$

$$C_7\left\{ess \sup_{t\in(0,T)} \int\limits_{\Omega} |\Phi^{(r)}(u_k)|^2 dx\right\}^{\frac{1}{p}+\frac{2}{n}-1} \int\limits_{Q_T} |\frac{\partial\Phi^{(r)}(u_k)}{\partial x}|^2 dxdt. \tag{1.11}$$

We choose $r_j = \frac{1}{2}\theta^{-j} - 1, j = 0, 1, \ldots,$ and obtain from (1.11)

$$\{I_k(r_j)\}^{\theta-1} \leq C_7^{\theta^{-j}}\theta^{-j}\{I_k(r_{j-1})\}^{\theta^{j-1}} \tag{1.12}$$

After iterating this estimate yields for arbitrary j the desired estimate. □

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