LOCAL SWITCHING OF SIMPLE FUSHIMI TREES

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ABSTRACT

When we treat with signed graphs corresponding to the root lattice A_n*, a connected graph is called a Fushimi tree if its all blocks are complete subgraphs. A Fushimi tree is said to be simple when by deleting any cut vertex we have its two connected components. Switching defines an equivalent relation in the set of all signed graphs. An equivalent class is called a switching class. Local switching partitions all signed graphs on* n *vertices into clusters of switching class. In this paper we have discussed about different sequences of local switching.*

Key words: ***Signed graph, Root lattice, Combinatorics.***

1. Introduction

For any base of the root lattice A_n we can construct a signed graph naturally. A signed graph is a graph whose edges are signed by +1 or –1. For a given signed graph Cameron, Seidel and Tsaranov constructed the corresponding root lattice [1]. In the present work we have treated with signed graphs corresponding to the root lattice A_n. Switching and local switching of signed graphs are introduced by Cameron, Goethals, Seidel and Shult [2]. A signed Fushimi tree is said to be a Fushimi tree with standard sign if it can be transformed by a switching to a signed Fushimi tree whose all edges are signed by +1. Here we have proved that any signed graph corresponding to A_n is a simple Fushimi tree with standard sign. Our main result is that a simple Fushimi tree with standard sign is contained in the cluster given by the line.

2. Definitions

2.1. Signed graph: A signed graph (G, f) is a graph G with a signing $f : E \rightarrow \{1, -1\}$ of the edges.

2.2. Local switching: Let S be a subset of the vertices of a graph G. Then the graph H formed by the vertices of S is called the local switching of G generated by S.

2.3. Fushmi tree: A connected graph is called a Fushmi tree if its all blocks are complete subgraphs.

3. Local switching

Let G be a simple Fushimi tree with standard sign. Take a block B of G. In the present work G is said to be $(n+k, k)$-type with respect to B if the order of B be $(n + k)$ and the number of cut vertices in B is k. Let a be a cut vertex in a block B. $G/\{a\}$ has two connected components. One is the component containing $B/\{a\}$. We call the other the branch with respect to (B, a). All such components are called branches with respect to B. A block of a simple Fushimi tree is said to be pendant if it has only one cut vertex. We call a simple Fushimi tree line-like if it has only one block or exactly two pendant blocks. A branch B_a with respect to (B, a) is called line-like if $B_a \cup \{a\}$ is line-like [3]. A simple Fushimi tree is said to be a line Fushimi tree if the order of its every block is 2.

Lemma 3.1. Let G be a simple Fushimi tree of $(n+k, k)$-type with respect to a block B. We can transform G into a simple Fushimi tree of (k, k)-type by a sequence of local switching.

Proof. Let the block B consists of vertices $a_1, a_2, a_3, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots, a_{n+k}$, where $a_{n+1}, a_{n+2}, a_{n+3}, \ldots, a_{n+k}$ are cut vertices. Set $J = \{a_{n+1}\}$ and $K = \{a_1, a_2, a_{n-1}, a_{n+2}, \ldots, a_{n+k}\}$. By local switching with respect to (a_n, J) we obtain a simple Fushimi tree of $(n+k–1, k)$-type with respect to block $\{a_1, a_2, a_3, \ldots, a_n, a_{n+2}, \ldots, a_{n+k}\}$, where $a_n, a_{n+2}, \ldots, a_{n+k}$ are cut vertices. Denote by G_1 this simple Fushimi tree of $(n+k–1, k)$-type. Applying the same procedure for G_1, we get a simple Fushimi tree of $(n+k–2, k)$-type. Repeating the same method, we get a simple Fushimi tree of (k, k)-type at last.

Lemma 3.2. Let G be a simple Fushimi tree of (k, k)-type with respect to a block B. Assume that all branches with respect to B are line-like. Then we can transform G into a simple Fushimi tree of $(k+n, k–1)$-type with respect to some block B, whose all branches are line-like, by a sequence of local switching, where n is some positive integer.

Proof. Let the block B consists of vertices $a_1, a_2, a_3, \ldots, a_k$. Take a branch, for example, B_1 with respect to (B, a_1). Assume the order of B_1 be n. Suppose $B_1 \cup \{a_1\}$ has m blocks. Firstly, assume $m = 1$. Take $i = a_1$, $J = B_1$, $K = \{a_2, \ldots, a_k\}$. Then by local switching with respect to

(a_1, B_1), G is transformed to a simple Fushimi tree of (k+n, k–1)-type with respect to the block {a_2, a_3, a_4, …, a_k} whose branches with respect to this block are all line-like. Suppose that the result is true for m. Assume that B_1 U {a_1} has (m + 1) blocks. Let C be the pendant block of B_1 U {a_1} and c its cut vertex. Put G_1 = {G\C} U {c}. Then G_1 is a simple Fushimi tree of (k, k)-type with respect to the block B. By the inductive hyposesis G_1 is transformed into a simple Fushimi tree G_2 of (k+n_1, k–1)-type with respect to the block {a_2, a_3, a_4, ..., a_k}, whose branches with respect to this block are all line-like, by a sequence of local switching, where $n_1 = n + 1 - n_2$ and n_2 is the order of the block C. By the same way we can transform G into G_2 U C, which is a special Fushimi tree of (k+n_1, k)-type with respect to the block {c, a_2, a_3, …, a_k} whose branches with respect to this block are all line-like and can be transformed into a simple Fushimi tree of (k+n, k–1)-type with respect to the block {a_2, a_3, a_4, …, a_k} having branches with respect to this block all line-like, by local switching.

Lemma 3.3. Let B be a simple Fushimi tree with one block. Then it can be transformed into a line Fushimi tree by a sequence of local switching.

Proof. Let B consists of vertices a_1, a_2, a_3, …, a_k. Set J = {a_1} and K = {a_3, a_4, a_5, …, a_k}. By local switching with respect to (a_2, J) we obtain a simple Fushimi tree of (k–1, 1)-type with respect to block {a_2, a_3, a_4, …, a_k}. Next, set J = {a_2} and K = {a_4, a_5, a_6, …, a_k}. By local switching with respect to (a_3, J) we obtain a simple Fushimi tree of (k–2, 1)-type with respect to block {a_3, a_4, a_5, …, a_k}. In this way we can get a line Fushimi tree by a sequence of local switching.

Lemma 3.4. Let G be a simple Fushimi tree of (n+k, k)-type with respect to a block B. Assume that all branches with respect to B are line-like. Then it can be transformed to a line Fushimi tree by a sequence of local switching. Especially a line-like special Fushimi tree can be transformed to a line Fushimi tree by a sequence of local switching [4].

Proof. By the same way in the proof of lemma 3.2, G can be transformed into a simple Fushimi tree G_1 of (k, k)-type with respect to some block B_1, by a sequence of local switching, whose branches with respect to B_1 are all line-like. By the same lemma we can get a simple Fushimi tree of $(k+n, k–1)$-type with some block B_2, whose branches are all line-like, by a sequence of local switching, where n is some positive integer. By a sequence of this process we obtain a simple Fushimi tree of $(k+N, 0)$, where N is some positive integer i.e. a simple Fushimi tree with one block, which is also transformed into a line Fushimi tree by lemma 3.3.

Lemma 3.5. Let G be a simple Fushimi tree. Then it has at least two pendant blocks.

Proof. If every block has more than one cut vertex, then, as we have no cycle, the order of the graph is infinite. Hence, it has at least two pendant blocks [5].

Theorem 3.1. Let G be a simple Fushimi tree. We can transform G into a line Fushimi tree by a sequence of local switching.

Proof. Assume G has m blocks. If $m = 1$, we get the result by Lemma 3.3. Suppose the result be true for $m = k$. Let $m = k + 1$. Take a pendant block B_1 of G with cut vertex b. Let B_2 be the other block with cut vertex b. Put $i = b$, $J = B_1\backslash b$ and $K = B_2\, B_1\backslash b$. By local switching with respect to (b, J) we obtain a simple Fushimi tree with k-blocks, which can be transformed into a line Fushimi tree by a sequence of local switching.

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