$K_{p,q}$ graph - Edge Product Number

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Abstract

A graph G is said to be an edge product graph if there exists an edge function f: E → P such that the function f and its corresponding edge product function F on V satisfies that F(v)∈P for every v∈V and if $e_1, e_2, \ldots, e_p \in E$ such that $f(e_1).f(e_2).\ldots.f(e_p) \in P$ then the edges $e_1, e_2, \ldots, e_p$ are incident on a vertex. In this paper the edge product number of complete bipartite graph is found.

Keywords: edge product function, edge product graph, edge product number of a graph

1. Introduction

We begin with simple, finite, connected and undirected graph G=(V(G),E(G)). For all other terminology and notations, follow Harary[1,2]. Here brief summary of definitions which are used for the present investigation are given. A dynamic survey on graph labeling is regularly updated by Gallian[3] and is published by electronic journal of combinatory.

2. Previous Results

Definition: 2.1

Let G be a graph and P be a set of positive integers with |E| = |P|. Then any bijection f: E → P is called an edge function of the graph G. Define F(v) = ∏{f(e)/e is incident on v}on V. Then F is called the edge product function of the edge function f. G is said to be an edge product graph if there exists an edge function f: E → P such that f and its corresponding edge product function F on V satisfy the conditions that F(v)∈P for every v∈V and if $e_1, e_2, \ldots, e_p \in E$ such that $f(e_1).f(e_2).\ldots.f(e_p) \in P$, then $e_1, e_2, \ldots, e_p$ are incident on a vertex.

Definition: 2.2

Let EPN(G), the edge product number of G, denote the minimum number K_2 components that must be added to G so that the resulting graph is an edge product graph. For any connected graph G other than K_2, EPN(G) ≥ 1. Let EPN(G) = r. An edge function f: E → P and its corresponding edge product function F which make $G \cup rK_2$ an edge product graph are respectively called an optimal edge function and an optimal edge product function of G.

Definition:2.3

Let $K_{(p,\ q)}$ be a complete bipartite graph with p+q vertices and pq edges. If $G=K_{(p,\ q)} \cup rK_2$ is a graph with $EPN(K_{p,\,q})=r$ where $r \geq 1$ and $2 \leq p \leq q$. Let the vertex set and the edge set of $K_{(p,\,q)}$ be $V_1=\{u_1,u_2,\ldots,u_p,v_1,v_2,\ldots,v_q\}$ and $E_1=\{u_iv_j:1\leq i\leq p \text{ and } 1\leq j\leq q\}$ respectively.

The mapping $f:E\to P$ is the optimal edge function and F is its corresponding optimal edge product function of f. The optimal edge function $f: E\to P$ can be restricted to the edge set E_1 and is represented by a matrix $(a_{i,j})$ of order $(p\times q)$, where $a_{i,\,j} = f(u_iv_j)$ for $1\leq i\leq p$ and $1\leq j\leq q$. Then the matrix $(a_{i,j})$ is known as edge function matrix of the complete bipartite graph $K_{(p,\,q)}$. The row product and the column product of the optimal edge product function F is defined by

$F(u_i) = i^{th}$ row product for $1 \leq i \leq p$ and

$F(v_j) = j^{th}$ column product for $1 \leq j \leq q$

Since u_i and v_j are the vertices for $1 \leq i \leq p$ and for $1 \leq j \leq q$. The i^{th} row elements are the labeling of the edges incident on u_i and the j^{th} column elements are the labeling of the edges incident on v_j.

Definition: 2.4

Let $U_{(p,q)}$ be a matrix of order $p\times q$. Then the two power unit product matrix $U_{(p,\,q)}$ is constructed by

$u_{(i,\,j)} = 2^{(i-1)+(j-1)(p-1)}$ for $1\leq i\leq(p-1)$ and for $1\leq j\leq(q-1)$

$u_{(p\,,j)} = \left(\dfrac{1}{2^{(j-1)(p-1)}\,\Pi\{2^{(k-1)}:\ 1\leq k\leq(p-1)}\right)$ for $1\leq j\leq(q-1)]$

$u_{(i,\,q)} = \left(\dfrac{1}{2^{(i-1)}\Pi\{2^{(k-1)(p-1)}:\ 1\leq k\leq(q-1)}\right)$ for $1\leq i\leq(p-1)]$

$u_{(p,\,q)} = \prod\{2^{k-1}: 1\leq k\leq(p-1)(q-1)\}$

3. Main Result

Lemma: 3.1

If $U_{(p,\,q)}$ is a two power unit product matrix of order $(p \times q)$. Then

(i) The product of row elements and column elements of the matrix $U_{(p,\,q)}$ are equal to one.

(ii) The product of a collection of distinct elements of $U_{(p,\,q)}$ are one then they form a row or a column.

Theorem: 3.2

If $K_{(p,\,q)}$ is a complete bipartite graph then $EPN(K_{p,\,q}) \geq 2$ for $2\leq p<q$.

Proof:

Assume that for a complete bipartite graph $K_{(p,\,q)}$, $EPN(K_{p,\,q})=1$ for some $2\leq p<q$. Consider a graph $G=K_{(p,\,q)}\cup K_2$ is an edge product graph for $2 \leq p < q$. Let $V=\{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q, w_1, w_2\}$ be the vertex set and $E=\{u_iv_j: 1\leq i\leq p \text{ and } 1\leq j\leq q\}\cup\{w_1w_2\}$ be the edge set of G. The mapping $f: E \to P$ is an optimal edge function and the function F is its corresponding optimal edge product function. Then F is an outer edge product function if $K_{(p,\,q)}$ has no pendent vertex and is triangle free. We have $F(u_i) = F(v_j) = f(w_1w_2 = x(\text{say})$, for $1\leq i\leq p$ and $1\leq j\leq q$.

Hence $\prod\{\prod\{f(u_iv_j):1\leq i\leq p: 1\leq j\leq q\} = \prod\{\prod\{f(u_iv_j): 1\leq j\leq q\}:1\leq i\leq p\}$.

Therefore, $\prod\{F(u_i):1\leq i\leq p\} = \prod\{F(v_j):1\leq j\leq q\}$.

That is, $x^q = x^p \Rightarrow q = p$ which is a contradiction to our hypothesis that q > p. Hence we have $EPN(K_{p, q}) \geq 2$ for $2 \leq p < q$.

Example:

If p = 2, q = 4 and then take $x = 2^5$ we have the edge function matrix A of order (2×4) is:

$$\begin{bmatrix} 2^{12} & 2^{13} & 2^{14} & 2^5 \\ 2^{10} & 2^9 & 2^8 & 2^{17} \end{bmatrix}$$

The graph $K_{(2, 4)} \cup 2K_2$ is an edge product graph and $EPN(K_{2, 4}) = 2$ is illustrated in the following figure.

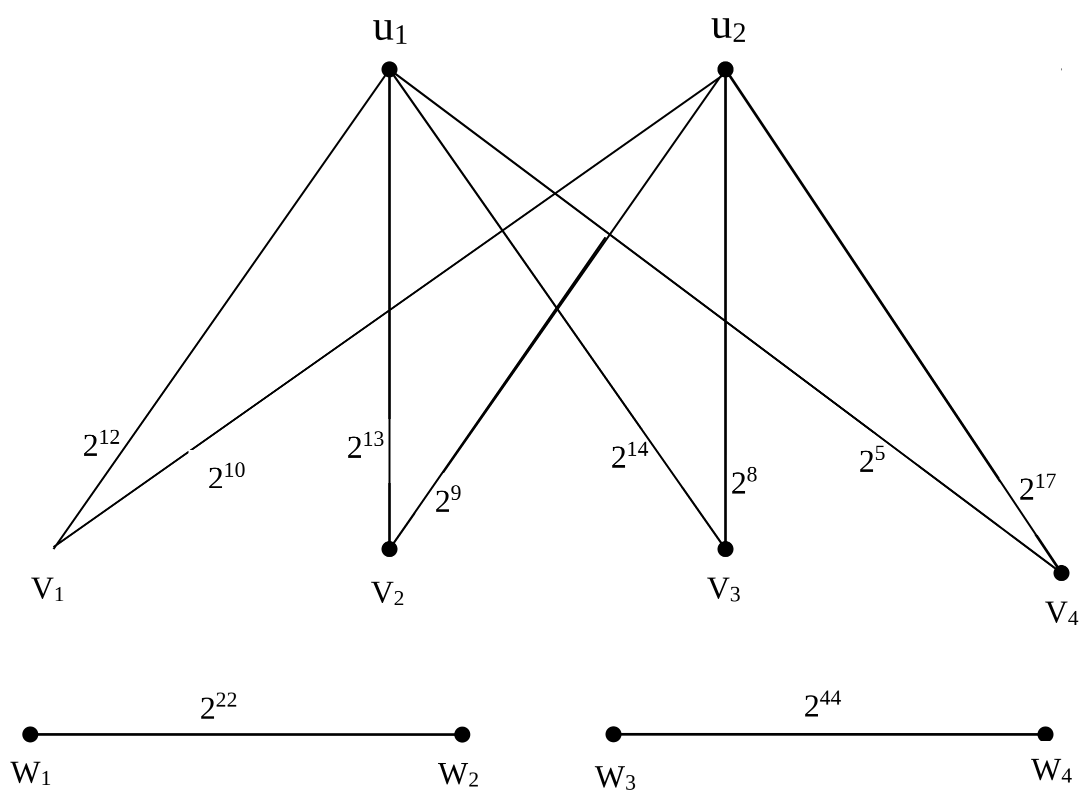

Figure 1

Theorem: 3.3

If $K_{(p, q)}$ is a complete bipartite graph for $2 \leq p < q$. Let $K_{(p, q)} \cup 2K_2$ be an edge product graph and $EPN(K_{(p, q)}) = 2$. Let $f: E \rightarrow P$ be an optimal edge function and the function F be its corresponding optimal edge product function of the graph G and the range of F={x, y} then [g.c.d of {x, y}] ∉{x, y}

Proof:

Consider a graph $K_{(p, q)} \cup 2K_2$ is an edge product graph for $2 \leq p < q$ with vertex set $V=\{u_1, u_2, \dots, u_p, v_1, v_2, \dots, v_q, w_1, w_2, w_3, w_4\}$ and edge set $E=\{u_iv_j: 1\leq i\leq p \text{ and } 1\leq j\leq q\}\cup\{w_1w_2, w_3w_4\}$.

$$\text{Let } F(u_i) = F(v_j) = \begin{cases} x \text{ for } 1 \le i \le p_1 \text{ and } 1 \le j \le q_1 \\ y \text{ for } (p_1+1) \le i \le p \text{ and } (q_1+1) \le j \le q \end{cases}$$

If $p_2 = (p / p_1)$ and $q_2 = (q / q_1)$ then $\prod\{F(u_i):1\le i\le p\} = \prod\{F(v_j):1\le i\le q\}$

Therefore $x^{p_1}y^{p_2} = x^{q_1}y^{q_2}$. If $p_1=q_1$ then $q_2 = p_2$ and $p = p_1p_2 = q_1q_2 = q$ which is a contradiction to our hypothesis that q>p.

If $p_1>q_1$ then clearly $q_2 > p_2$. Consider [g.c.d of {x, y}] = y. Then y is a divisor of x.

Let $x = y^p$, then $y^{p(p1-q1)} = y^{(q2-p2)} \Rightarrow p = [(q_2/p_2) / (p_1/q_1)] \le (q_2/p_2) \le q_2$.

In one partition there are at least p vertices whose image under the function F which are equal to y, taking the entire edges incident on them we get that their total labeling is equal to x. But they are all not incident on a vertex.

This is a contradiction that the graph G is an edge product graph. Hence y is not a divisor of x. Similarly x is not a divisor of y.

Therefore [g.c.d of {x, y}] is neither x nor y. Thus we obtain that [g.c.d of {x, y}]$\notin${x, y}.

References

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[2] F. Harary, *Sum graphs over all the integers*, Discrete Math., 124 (1994) 99-105

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