An M/G/1 Retrial Queue with Unreliable Server under Fuzzy Environment

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Abstract

An M/G/1 retrial queueing system with breakdowns have been studied in fuzzy environment. The arrival rate,retrial rate, service rate,failure rate and repair rate of server are all fuzzy numbers. For this model we obtain some system characteristics such as mean orbit size ,mean normal queue size and mean system size. The α cut approach is used to transform fuzzy queues with an unreliable server to a family of crisp retrial queues with unreliable server. By means of the membership functions of the system characteristics, a set of parametric nonlinear programme is developed to describe the family of crisp queues with an unreliable server.Numerical example is also illustrated to test the feasibility.

Keywords: Retrial queue, Breakdowns, Parametric Programming, Membership functions
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1 Introduction

Queueing models have wider applications in service organizations as well as manufacturing firms, in that various types of customers are serviced by various types of servers according to specific queue discipline [4]. However, in many real-life situations the server may experience unpredictable breakdowns. Therefore, queueing models with server breakdowns provide a realistic representation of such systems.

In traditional queueing theory, the inter arrival times, service times and inter retrial times are assumed to follow certain probability distributions with fixed parameters. But, in real life in many situations the parameter may only be characterized subjectively,that is, the system parameters are both possibilistic and probabilistic. Thus fuzzy analysis would be potentially much more useful and realistic than the commonly used crisp concepts.

Li and Lee [10] investigated analytical results for two fuzzy queues using a general approach based on Zadeh's extension principle. Negi and Lee [14] proposed a procedure using α -cut and two variable simulation to analyze fuzzy queues. Using parametric programming Kao et al. [11] constructed the membership functions of system characteristics for fuzzy queues. A queueing model with unreliable server under fuzzy environment done by Ke.J.C et.al [19]. Santhakumaran.A and Shanmugasundaram.S [7] proposed a Single server retrial queue in Bernoulli schedule with feedback on non-retrial Customers. Shanmugasundaram.S and Venkatesh.B [21] discussed multi-server fuzzy queueing model using DSW algorithm.

2 Description of the system

The basic queueing model of this paper is M/G/1 retrial queue with unreliable server. Customers join the retrial orbit if and only if they are interrupted by server breakdown. Retrial customers do not join the normal queue, but rather attempt to access the server directly at random intervals independently of arrivals or other

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retrial customers.However these interrupted customers can access to the server only when it is operational and idle and repeat service until they have been successfully processed.We allow for both active breakdowns which occur during a service cycle and idle breakdowns which occur while the server is not failed but idle. The server may not breakdown while under repair.

In this work, we have used five fuzzy variables, namely the fuzzified exponential arrival rate,retrial rate,service rate,failure rate and repair rate. Through the α cuts and Zadeh's extension principle [9] ,we transform fuzzy queues with unreliable server to a family of crisp retrial queues with unreliable server. As α value varies,the family of crisp queues are then described and solved by parametric nonlinear programming (NLP).The solutions from NLP completely and successfully derive the membership functions of the system characteristics.The remainder of this paper is composed as follows: Section 2 describe the basic queueing model and section 3 gives and crisp queue results. In section 4 ,a mathematical programming approach is discussed for deriving the membership functions of these system characteristics. A numerical example is given in section 5. Conclusions are drawn in section 6.

3 The crisp model

Customers arrive to the system according to a poisson process with rate λ ¿0 and the arriving customers form a waiting line of infinite capacity infront of an unreliable server.The retrial rate is θ ,the service rate is γ the failure rate of server is σ and the repair rate is β. We consider a queueing model M/G/1 retrial queue with unreliable server.The system characteristics of interest are mean orbit size E[R],normal queue size E[Q] and system size E[N].

The queueing system is stable if and only if $\rho < 1$:

$$\rho = \frac{\lambda(1-b^*(\sigma))(\sigma+\beta)}{\beta b^*(\sigma)\sigma}$$

(i) Probability that the server is idle (p_I)

$$p_I = \frac{\beta}{\beta+\sigma} - \frac{\lambda(1-b^*(\sigma))}{\sigma b^*(\sigma)} \tag{3.1}$$

(ii) Probability that the server is failure (p_F)

$$p_F = \frac{\sigma}{\beta+\sigma} \tag{3.2}$$

(iii)Probability that the server is busy (p_B)

$$p_B = \frac{\lambda(1-b^*(\sigma))}{\sigma b^*(\sigma)} \tag{3.3}$$

$$E[R] = \frac{\rho}{1-\rho}[\frac{\beta}{\beta+\sigma}\frac{\sigma b^*(\sigma)[\sigma-\lambda(1-b^*(\sigma))]+(\beta+\sigma)[\lambda(1-b^*(\sigma))-\sigma\hat{B}']}{b^*(\sigma)[\beta\sigma-\lambda(1-b^*(\sigma))(\beta+\sigma)]}+\frac{\sigma}{\theta}] \tag{3.4}$$

$$E[Q] = \lambda\frac{\sigma^3 b^*(\sigma)-(\beta+\sigma)^2[\sigma\hat{B}'-\lambda(1-b^*(\sigma))]}{\sigma b^*(\sigma)(\beta+\sigma)[\beta\sigma-\lambda(1-b^*(\sigma))(\beta+\sigma)]} \tag{3.5}$$

$$E[N] = \frac{\lambda b^*(\sigma)\sigma^3+(1-b^*(\sigma))[\beta\sigma(\beta+2\sigma)+\lambda(\beta+\sigma)^2]-\lambda\sigma(\beta+\sigma)^2\hat{B}'}{\sigma b^*(\sigma)(\beta+\sigma)[\beta b^*(\sigma)-\lambda(1-b^*(\sigma))(\beta+\sigma)]}+\frac{\sigma\rho}{\theta(1-\rho)} \tag{3.6}$$

Particular Case: We assume that service times are uniformly distributed on the interval $(0,\frac{2}{\mu})$ Then $A_1 = b^*(\sigma) = \gamma\frac{1-exp(\frac{-2\sigma}{\gamma})}{2\sigma}$ $A_2 = \hat{B}' = \lambda\gamma\frac{1-exp(\frac{-2\sigma}{\gamma})(1+\frac{2\sigma}{\gamma})}{2\sigma^2}$

4 Fuzzy retrial queue with an unreliable server

To ensure that the above system has wider applications, we extend it to the fuzzy environment. Suppose the arrival rate λ,retrial rate θ service rateγ , failure rate σ ,repair rate β are approximately known and can be represented by the fuzzy sets λ,θ,γ,σ,β respectively.

Let $\mu_{\tilde{\lambda}}(x), \mu_{\tilde{\theta}}(r), \mu_{\tilde{\gamma}}(s), \mu_{\tilde{\sigma}}(u), \mu_{\tilde{\beta}}(v)$ denote the membership functions of λ, θ,γ,σ,β respectively. Then we have the following fuzzy sets.

$$\tilde{\lambda} = \{x, \mu_{\tilde{\lambda}}(x), x\epsilon X\}$$

$$\tilde{\theta} = \{r, \mu_{\tilde{\theta}}(r), r\epsilon R\}$$

$$\tilde{\gamma} = \{s, \mu_{\tilde{\gamma}}(s), s\epsilon S\}$$

$$\tilde{\sigma} = \{u, \mu_{\tilde{\sigma}}(u), u\epsilon U\}$$

$$\tilde{\beta} = \{v, \mu_{\tilde{\beta}}(v), v\epsilon V\}$$

where X,R,S,U,V are crisp universal sets of arrival rate,retrial rate,service rate,failure rate,repair rate respectively. Let f(x,r,s,u,v) denote the system characteristics of interest. Since λ,θ,γ,σ,β are fuzzy numbers, f(λ,θ,γ,σ,β) is also a fuzzy number. Based on Zadeh's extension principle [9],the membership function of the system characteristic f($\tilde{\lambda}$,$\tilde{\theta}$,$\tilde{\gamma}$,$\tilde{\sigma}$,$\tilde{\beta}$) is defined as

$$\mu_{f(\tilde{\lambda},\tilde{\theta},\tilde{\gamma},\tilde{\sigma},\tilde{\beta})} = \sup_{x\in X, r\in R, s\in S, u\in U, v\in V} \{min\,\{\mu_{\tilde{\lambda}}(x),\ \mu_{\tilde{\theta}}(r)\ , \mu_{\tilde{\gamma}}(s),\ \mu_{\tilde{\sigma}}(u),\ \mu_{\tilde{\beta}}(v)\ / z = f(x, r, s, u, v)\} \tag{4.1}$$

The general service time distribution consider with two service times are exponentially distributed with parameter μ and uniformly distributed on the interval $(0, \frac{2}{\gamma})$ Particular Case: The service times are uniformly distributed then $b^*(\sigma)$ and $\hat{B}^{'}$ given by $A_1 = b^*(\sigma) = \frac{s}{2\gamma}[1 - exp(\frac{-2u}{s})]$ $A_2 = \hat{B}^{'} = \frac{xs}{2u^2}[1 - exp(\frac{-2u}{s})(1 + \frac{2u}{s})]$ $A_3 = \frac{x(1-A_1)(u+v)}{uvA_1} = \rho$ The system characteristic of interest are expected number of customers in orbit E[R],mean normal queue size E[Q], and expected number of customers in the system E[N].

From equations (3.1) and (4.1)the membership function of p_I is

$$\mu_{\tilde{p_I}}(z) = \sup_{x\in X, r\in R, s\in S, u\in U, v\in V} \{min\,\{\mu_{\tilde{\lambda}}(x),\ \mu_{\tilde{\theta}}(r)\ , \mu_{\tilde{\gamma}}(s),\ \mu_{\tilde{\sigma}}(u),\ \mu_{\tilde{\beta}}(v)/z\}\} \tag{4.2}$$

where

$$z = \frac{v}{u+v} - \frac{x(1-A_1)}{uA_1}$$

similarly from equations (3.2) and (4.1),the membership function of p_F is

$$\mu_{\tilde{p_F}}(z) = \sup_{x\in X, r\in R, s\in S, u\in U, v\in V} \{min\{\mu_{\tilde{\lambda}}(x),\ \mu_{\tilde{\theta}}(r), \mu_{\tilde{\gamma}}(s), \mu_{\tilde{\sigma}}(u), \mu_{\tilde{\beta}}(v)\ /z\}\} \tag{4.3}$$

where

$$z = \frac{u}{x+u}$$

Now from equations (3.3) and (4.1), the membership function of p_B is

$$\mu_{\tilde{p_B}}(z) = \sup_{x\in X, r\in R, s\in S, u\in U, v\in V} \{min\{\mu_{\tilde{\lambda}}(x),\ \mu_{\tilde{\theta}}(r), \mu_{\tilde{\gamma}}(s), \mu_{\tilde{\sigma}}(u), \mu_{\tilde{\beta}}(v)\ /z\}\} \tag{4.4}$$

where

$$z = \frac{x(1-A_1)}{uA_1}$$

From (3.4) and (4.1),the membership function of E[R] is

$$\mu_{\tilde{E[R]}}(z_1) = \sup_{x\in X, r\in R, s\in S, u\in U, v\in V} \{min\,\{\mu_{\tilde{\lambda}}(x),\ \mu_{\tilde{\theta}}(r)\ , \mu_{\tilde{\gamma}}(s),\ \mu_{\tilde{\sigma}}(u),\ \mu_{\tilde{\beta}}(v)\ /z_1\}\} \tag{4.5}$$

where

$$z_1 = (\frac{A_3}{1-A_3})\frac{(\frac{uv}{u+v})A_1[u-x(1-A_1)]+(u+v)[x(1-A_1-uA_2)]}{A_1[uv-x(1-A_1)(u+v)]}+\frac{u}{y}$$

From (3.5) and (4.1),the membership function of E[Q] is

$$\mu_{E[\tilde{Q}]}(z_2) = \sup_{x\in X,r\in R,s\in S,u\in U,v\in V} \{min\,\{\mu_{\tilde{\lambda}}(x),\ \mu_{\tilde{\theta}}(r)\ ,\mu_{\tilde{\gamma}}(s),\ \mu_{\tilde{\sigma}}(u),\ \mu_{\tilde{\beta}}(v)\ /z_2\}\} \tag{4.6}$$

where

$$z_2 = \frac{xu^3A_1-(u+v)^2[uA_2-x(1-A_1)]}{uA_1(u+v)[uv-x(1-A_1)(u+v)]}$$

From equations (3.6) and (4.1), the membership function of E[N] is

$$\mu_{E[\tilde{N}]}(z_3) = \sup_{x\in X,r\in R,s\in S,u\in U,v\in V} \{min\{\mu_{\tilde{\lambda}}(x),\ \mu_{\tilde{\theta}}(r),\mu_{\tilde{\gamma}}(s),\mu_{\tilde{\sigma}}(u),\mu_{\tilde{\beta}}(v)\ /z_3\}\} \tag{4.7}$$

where

$$z_3 = \frac{xA_1[(1-A_1)uv(v+2u)+x(u+v)^2]-xu(u+v)^2A_2}{uA1(u+v)[uA_1-x(1-A1)(u+v)]}+\frac{uA_3}{y(1-A_3)}$$

The membership functions in (4.2),(4.3),(4.4),(4.5),(4.6),(4.7) are not in the usual forms for practical use and making it very difficult to imagine their shapes.

In this paper we approach the problem using a mathematical programming technique. These parametric nonlinear programs are developed to find the α cuts of f(λ,θ,γ,σ,β) based on the extension principle.

5 The parametric nonlinear programming approach

One approach is to construct the membership function $\mu_{f(\tilde{\lambda},\tilde{\theta},\tilde{\gamma},\tilde{\sigma},\tilde{\beta})}$ by deriving the α-cuts of $\mu_{f(\tilde{\lambda},\tilde{\theta},\tilde{\gamma},\tilde{\sigma},\tilde{\beta})}$ The α-cuts of λ,θ,γ,σ,β are defined respectively as follows.

$$\lambda(\alpha) = \{x\epsilon X/\mu_{\tilde{\lambda}}(x) \geq \alpha\} \tag{5.1}$$

$$\theta(\alpha) = \{r\epsilon R/\mu_{\tilde{\theta}}(r) \geq \alpha\} \tag{5.2}$$

$$\gamma(\alpha) = \{s\epsilon S/\mu_{\tilde{\gamma}}(s) \geq \alpha\} \tag{5.3}$$

$$\sigma(\alpha) = \{u\epsilon U/\mu_{\tilde{\sigma}}(u) \geq \alpha\} \tag{5.4}$$

$$\beta(\alpha) = \{v\epsilon V/\mu_{\tilde{\beta}}(v) \geq \alpha\} \tag{5.5}$$

The fuzzy arrival rate $\tilde{\lambda}$, fuzzy retrial rate $\tilde{\theta}$,fuzzy service rate $\tilde{\gamma}$,fuzzy failure rate $\tilde{\sigma}$, fuzzy repair rate $\tilde{\beta}$ of the queueing system are fuzzy numbers. Therefore the α level sets of $\tilde{\lambda},\tilde{\theta},\tilde{\gamma},\tilde{\sigma},\tilde{\beta}$ defined in equations (5.1 - 5.5) are crisp intervals which can be expressed in the following forms.

$$\lambda(\alpha) = [x_L, x_U] = [\ \min_{x\in X}\{x/\mu_{\tilde{\lambda}}(x) \geq \alpha\},\ \max_{x\in X}\{x/\mu_{\tilde{\lambda}}(x) \geq \alpha\}\ ] \tag{5.6}$$

$$\theta(\alpha) = [r_L, r_U] = [\ \min_{r\in R}\{r/\mu_{\tilde{\theta}}(r) \geq \alpha\},\ \max_{r\in R}\{r/\mu_{\tilde{\theta}}(r) \geq \alpha\}\ ] \tag{5.7}$$

$$\gamma(\alpha) = [s_L, s_U] = [\ \min_{s\in S}\{s/\mu_{\tilde{\gamma}}(s) \geq \alpha\},\ \max_{s\in S}\{s/\mu_{\tilde{\gamma}}(s) \geq \alpha\}\ ] \tag{5.8}$$

$$\sigma(\alpha) = [u_L, u_U] = [\ \min_{u\in U}\{u/\mu_{\tilde{\sigma}}(u) \geq \alpha\},\ \max_{u\in U}\{u/\mu_{\tilde{\sigma}}(u) \geq \alpha\}\ ] \tag{5.9}$$

$$\beta(\alpha) = [v_L, v_U] = [\ \min_{v\in V}\{v/\mu_{\tilde{\beta}}(v) \geq \alpha\},\ \max_{v\in V}\{v/\mu_{\tilde{\beta}}(v) \geq \alpha\}\ ] \tag{5.10}$$

Fuzzy queues reduces to a family of crisp queues with deterministic inter arrival time, retrial time,service time, failure rate and repair rate for different α level sets. $\{\lambda(\alpha), 0<\alpha\leq 1\}, \{\theta(\alpha), 0<\alpha\leq 1\}$, $\{\gamma(\alpha), 0<\alpha\leq 1\}, \{\sigma(\alpha), 0<\alpha\leq 1\}, \{\beta(\alpha), 0<\alpha\leq 1\}$

$$x_\alpha{}^L = \min\ \mu_{\tilde{\lambda}}^{-1}(\alpha), x_\alpha{}^V = \max\ \mu_{\tilde{\lambda}}^{-1}(\alpha) \tag{5.11}$$

$$r_\alpha{}^L = \min\ \mu_{\tilde{\theta}}^{-1}(\alpha), r_\alpha{}^V = \max\ \mu_{\tilde{\theta}}^{-1}(\alpha) \tag{5.12}$$

$$s_\alpha{}^L = \min\ \mu_{\tilde{\gamma}}^{-1}(\alpha), s_\alpha{}^V = \max\ \mu_{\tilde{\gamma}}^{-1}(\alpha) \tag{5.13}$$

$$u_\alpha{}^L = \min\ \mu_{\tilde{\sigma}}^{-1}(\alpha), u_\alpha{}^V = \max\ \mu_{\tilde{\sigma}}^{-1}(\alpha) \tag{5.14}$$

$$v_\alpha{}^L = \min\ \mu_{\tilde{\beta}}^{-1}(\alpha), v_\alpha{}^V = \max\ \mu_{\tilde{\beta}}^{-1}(\alpha) \tag{5.15}$$

The α cut approach can be used to develop the membership functions. Based on Zadeh's extension principle $\mu_{\tilde{A}}(I)$ is the supremum and minimum over
$\{\mu_{\tilde{\lambda}}(x),\ \mu_{\tilde{\theta}}(r)\ , \mu_{\tilde{\gamma}}(s),\ \mu_{\tilde{\sigma}}(u),\ \mu_{\tilde{\beta}}(v)\}$
$\tilde{A}$ is any performance measures of interest and z=f(x,r,s,u,v) satisfying $\mu_{\tilde{A}}(z)=\alpha,\ 0<\alpha\leq 1$ The following five cases :

case (i): $(\mu_{\tilde{\lambda}}(x)=\alpha, \mu_{\tilde{\theta}}(r)\geq\alpha, \mu_{\tilde{\gamma}}(s)\geq\alpha, \mu_{\tilde{\sigma}}(u)\geq\alpha, \mu_{\tilde{\beta}}(v)\geq\alpha)$

case (ii): $(\mu_{\tilde{\lambda}}(x)\geq\alpha, \mu_{\tilde{\theta}}(r)=\alpha, \mu_{\tilde{\gamma}}(s)\geq\alpha, \mu_{\tilde{\sigma}}(u)\geq\alpha, \mu_{\tilde{\beta}}(v)\geq\alpha)$

case (iii): $(\mu_{\tilde{\lambda}}(x)\geq\alpha, \mu_{\tilde{\theta}}(r)\geq\alpha, \mu_{\tilde{\gamma}}(s)=\alpha, \mu_{\tilde{\sigma}}(u)\geq\alpha, \mu_{\tilde{\beta}}(v)\geq\alpha)$

case (iv): $(\mu_{\tilde{\lambda}}(x)\geq\alpha, \mu_{\tilde{\theta}}(r)\geq\alpha, \mu_{\tilde{\gamma}}(s)\geq\alpha, \mu_{\tilde{\sigma}}(u)=\alpha, \mu_{\tilde{\beta}}(v)\geq\alpha)$

case (v): $(\mu_{\tilde{\lambda}}(x)\geq\alpha, \mu_{\tilde{\theta}}(r)\geq\alpha, \mu_{\tilde{\gamma}}(s)\geq\alpha, \mu_{\tilde{\sigma}}(u)\geq\alpha, \mu_{\tilde{\beta}}(v)=\alpha)$

The non-linear programming technique gives the lower and upper bounds of α cuts $\mu_{E[\tilde{R}](Z_1)}$ for case(i) as

$$(E[R])_L = \min\,(\frac{A_3}{1-A_3})\frac{(\frac{uv}{u+v})A_1[u-x(1-A_1)]+(u+v)[x(1-A_1-uA_2)]}{A_1[uv-x(1-A_1)(u+v)]}+\frac{u}{y}$$

$$(E[R])_U = \max\,(\frac{A_3}{1-A_3})\frac{(\frac{uv}{u+v})A_1[u-x(1-A_1)]+(u+v)[x(1-A_1-uA_2)]}{A_1[uv-x(1-A_1)(u+v)]}+\frac{u}{y}$$

such that the equation $\mu_{E[\tilde{R}]}(z_1)=\alpha$ is true only when either one of the above cases obtained from equations () and () represent the α-cut of E[R]. If both $(E[R])_L$ and $(E[R])_U$ in equations (),() are invertible with respect to α , then the left shape function $L[z_1]=((E[R])_L)^{-1}$

Similarly we calculate the lower and upper bounds of α cuts of $\mu_{\tilde{A}}$ for case (ii),(iii),(iv),(v) such that $x_\alpha^L\leq x\leq x_\alpha^U,\ y_\alpha^L\leq y\leq y_\alpha^U$ The crisp interval [$(z)_\alpha^L$ $(z)_\alpha^U$] represents the α cuts of $\tilde{z}$, both $(z)_\alpha^L$ and $(z)_\alpha^U$ are invertible w.r.t α then left shape function $L(z)=(I_L)^{-1}$ and right shape function $R(z)=(I_U)^{-1}$ $\mu_{\tilde{A}}(z)$ can be written as $\mu_{\tilde{A}}(z)=\begin{cases} L(z), & \text{if } (A)_{\alpha=0}^L\leq I\leq (A)_{\alpha=1}^L\ ; \\ 1, & \text{if } (A)_{\alpha=1}^L\leq I\leq (A)_{\alpha=1}^U\ ; \\ R(z), & \text{if } (A)_{\alpha=1}^U\leq I\leq (A)_{\alpha=0}^L. \end{cases}$ Using the above technique the idle probability , failure probability ,busy probability given by the lower and upper bounds of α cuts for $\mu_{\tilde{p_I}}, \mu_{\tilde{p_F}}, \mu_{\tilde{p_B}}$ are

$$(\mu_{\tilde{p_I}})_\alpha^L = \min\{\frac{v}{u+v}-\frac{x(1-A_1)}{uA_1}\}$$

$$(\mu_{\tilde{p_I}})_\alpha^U = \max\{\frac{v}{u+v}-\frac{x(1-A_1)}{uA_1}\}$$

$$(\mu_{\tilde{p_F}})_\alpha^L = \min\{\frac{u}{u+x}\}$$

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$$(\mu_{\tilde{p_F}})_\alpha^U = \max\{\frac{u}{u+x}\}$$

$$(\mu_{\tilde{p_B}})_\alpha^L = \min\{\frac{x(1-A_1)}{uA_1}\}$$

$$(\mu_{\tilde{p_B}})_\alpha^U = \max\{\frac{x(1-A_1)}{uA_1}\}$$

with $x_\alpha^L \leq x \leq x_\alpha^U,\ r_\alpha^L \leq r \leq\ r_\alpha^U,\ s_\alpha^L \leq s \leq s_\alpha^U,\ u_\alpha^L \leq u \leq u_\alpha^U,\ v_\alpha^L \leq v \leq v_\alpha^U$

From which the membership functions of $\mu_{\tilde{p_I}}(z), \mu_{\tilde{p_{FI}}}(z), \mu_{\tilde{p_B}}(z)$ can be constructed as

$$\mu_{\tilde{p_I}}(z) = \begin{cases} L(z), & \text{if } (L_{\tilde{p_I}})_{\alpha=0}^L \leq z \leq (L_{\tilde{p_I}})_{\alpha=1}^L\ ; \\ 1, & \text{if } (L_{\tilde{p_I}})_{\alpha=1}^L \leq z \leq (L_{\tilde{p_I}})_{\alpha=1}^U\ ; \\ R(z), & \text{if } (L_{\tilde{p_I}})_{\alpha=1}^U \leq z \leq (L_{\tilde{p_I}})_{\alpha=0}^L. \end{cases}$$

$$\mu_{\tilde{p_F}}(z) = \begin{cases} L(z), & \text{if } (L_{\tilde{p_F}})_{\alpha=0}^L \leq z \leq (L_{\tilde{p_F}})_{\alpha=1}^L\ ; \\ 1, & \text{if } (L_{\tilde{p_F}})_{\alpha=1}^L \leq z \leq (L_{\tilde{p_F}})_{\alpha=1}^U\ ; \\ R(z), & \text{if } (L_{\tilde{p_F}})_{\alpha=1}^U \leq z \leq (L_{\tilde{p_F}})_{\alpha=0}^L. \end{cases}$$

$$\mu_{\tilde{p_B}}(z) = \begin{cases} L(z), & \text{if } (L_{\tilde{p_B}})_{\alpha=0}^L \leq z \leq (L_{\tilde{p_B}})_{\alpha=1}^L\ ; \\ 1, & \text{if } (L_{\tilde{p_B}})_{\alpha=1}^L \leq z \leq (L_{\tilde{p_B}})_{\alpha=1}^U\ ; \\ R(z), & \text{if } (L_{\tilde{p_B}})_{\alpha=1}^U \leq z \leq (L_{\tilde{p_B}})_{\alpha=0}^L. \end{cases}$$

The membership functions of $\mu_{\tilde{E[R]}}, \mu_{\tilde{E[Q]}}, \mu_{\tilde{E[N]}}$ can be constructed as

$$\mu_{\tilde{E[R]}}(z_1) = \begin{cases} L(z_1), & \text{if } (L_{\tilde{E[R]}})_{\alpha=0}^L \leq (z_1) \leq (L_{\tilde{E[R]}})_{\alpha=1}^L\ ; \\ 1, & \text{if } (L_{\tilde{E[R]}})_{\alpha=1}^L \leq (z_1) \leq (L_{\tilde{E[R]}})_{\alpha=1}^U\ ; \\ R(z_1), & \text{if } (L_{\tilde{E[R]}})_{\alpha=1}^U \leq (z_1) \leq (L_{\tilde{E}[R]})_{\alpha=0}^L. \end{cases}$$

$$\mu_{\tilde{E[Q]}}(z_1) = \begin{cases} L(z_2), & \text{if } (L_{\tilde{E[Q]}})_{\alpha=0}^L \leq (z_2) \leq (L_{\tilde{E[Q]}})_{\alpha=1}^L\ ; \\ 1, & \text{if } (L_{\tilde{E[Q]}})_{\alpha=1}^L \leq (z_2) \leq (L_{\tilde{E[Q]}})_{\alpha=1}^U\ ; \\ R(z_2), & \text{if } (L_{\tilde{E[Q]}})_{\alpha=1}^U \leq (z_2) \leq (L_{\tilde{E}[Q]})_{\alpha=0}^L. \end{cases}$$

$$\mu_{\tilde{E[N]}}(z_3) = \begin{cases} L(z_3), & \text{if } (L_{\tilde{E[N]}})_{\alpha=0}^L \leq (z_3) \leq (L_{\tilde{E[N]}})_{\alpha=1}^L\ ; \\ 1, & \text{if } (L_{\tilde{E[N]}})_{\alpha=1}^L \leq (z_3) \leq (L_{\tilde{E[N]}})_{\alpha=1}^U\ ; \\ R(z_3), & \text{if } (L_{\tilde{E[N]}})_{\alpha=1}^U \leq (z_3) \leq (L_{\tilde{E}[N]})_{\alpha=0}^L. \end{cases}$$

6 Numerical example

If the arrival rate λ, the retrial rate θ,the service rate γ,the failure rate σ, the repair rate β are trapezoidal fuzzy numbers per unit time described by $\tilde{\lambda} = [3,4,5,6], \tilde{\theta} = [14,15,16,17], \tilde{\gamma} = [25,26,27,28], \tilde{\sigma} = [36,37,38,39], \tilde{\beta} = [46,47,48,49]$. $\mu_{\tilde{A}}(z)$ can be written as $\mu_{\tilde{A}}(z) = \begin{cases} L(z), & \text{if } (A)_{\alpha=0}^L \leq I \leq (A)_{\alpha=1}^L\ ; \\ 1, & \text{if } (A)_{\alpha=1}^L \leq I \leq (A)_{\alpha=1}^U\ ; \\ R(z), & \text{if } (A)_{\alpha=1}^U \leq I \leq (A)_{\alpha=0}^L. \end{cases}$

With the help of Matlab , we perform α - cuts of arrival rate and service rate and fuzzy expected number of customers in queue at eleven distinct α levels: 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1. Crisp intervals for fuzzy expected number of customers in orbit (E[R]) at different possibilistic levels are presented in table. Similarly other performance measure such as expected number of customers in queue (E[Q]), number of customers in system (E[N]) also derived in the table.

The α - cut represent the possibility that these four performance measure will lie in the associated range. Specially, $\alpha = 0$ the range, the performance measures could appear and for $\alpha = 1$ the range,the performance measures are likely to be.

For example, while these four performance measures are fuzzy, the most likely value of E[R] falls between 1.8317 and 4.0525 and its value is impossible to fall outside the range of 0.9462 and 11.1282; it is definitely possible that the expected number of customers in queue E[Q] falls between 0.1323 and 0.2220, and it will never fall below 0.0767 and exceed 0.3676 ;expected number of customers in the system E[N] falls between 2.3138 and 5.2849, and it will never fall below 1.1930 or exceed 24.6253. The above information will be very useful for designing a queueing system.

Table 1: The α-cuts of E[R],E[Q],E[N]

α	E[R]	E[Q]	E[N]
0.0	[0.9462 , 11.1282]	[0.0767 , 0.3676]	[1.1930 , 24.6253]
0.1	[1.0067 , 9.9279]	[0.0812 , 0.3496]	[1.2705 , 19.0044]
0.2	[1.0719 , 8.8870]	[0.0858 , 0.3325]	[1.3537 , 15.3592]
0.3	[1.1422 , 7.9805]	[0.0907 , 0.3162]	[1.4431 , 12.7951]
0.4	[1.2181 , 7.1878]	[0.0959 , 0.3007]	[1.5395 , 10.8910]
0.5	[1.3003 , 6.4922]	[0.1012 , 0.2859]	[1.6436 , 9.4211]
0.6	[1.3894 , 5.8796]	[0.1069 , 0.2719]	[1.7562 , 8.2524]
0.7	[1.4862 , 5.3384]	[0.1128 , 0.2585]	[1.8783 , 7.3015]
0.8	[1.5915 , 4.8588]	[0.1190 , 0.2457]	[2.0110 , 6.5135]
0.9	[1.7063 , 4.4324]	[0.1255 , 0.2336]	[2.1557 , 5.8502]
1.0	[1.8317 , 4.0525]	[0.1323 , 0.2220]	[2.3138 , 5.2849]

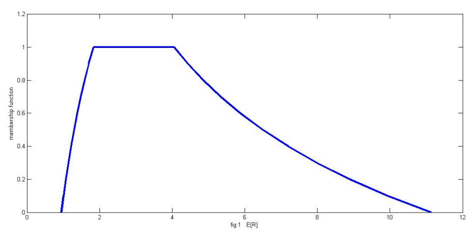

Figure 1: Expected number of customers in orbit E[R]

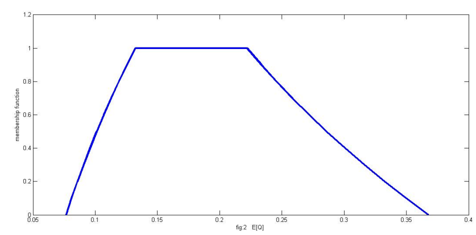

Figure 2: Expected number of customers in queue E[Q]

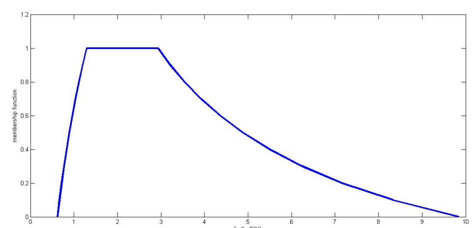

Figure 3: Expected number of customers in system E[N]

7 Conclusion

This paper applied the concept of α-cuts and Zadeh's extension principle to transform a fuzzy queue with an unreliable server into a family of crisp queues that can be described by a set of parametric nonlinear programs(NLP).Due to the complexity of four fuzzy parameters,the closed form for the corresponding membership function can not be explicitly derived by taking the inverse of its α cuts at different possibility levels. Numerical solutions for different α values were calculated to approximate the membership functions by NLP. These results are significant as well as useful for system designers. Numerical example illustrated that the curve is trapezoidal form.

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