# *CUBIC SOFT MATRICES*

V. Chinnadurai[a*] and S. Barvkavi[a]

[a]*Department of Mathematics, Annamalai University, Annamalainagar- 608002, Tamilnadu, India.*
Email:kv.chinnadurai@yahoo.com and barkavi79@gmail.com

In this paper, we introduce the notion of cubic soft matrices and the concepts of internal and external cubic soft matrices, P-(R)-order, P-(R)-union, P-(R)-intersection, P-OR, P-AND, R-OR and R-AND of cubic soft matrices and their related properties have also been investigated.

**Keywords:** cubic soft sets, P-order cubic soft sets, R-order cubic soft sets, fuzzy matrices, cubic soft matrices.

## 1. Introduction

Fuzzy set theory was proposed by Lotifi A. Zadeh [11] and it has extensive applications in various fields. In 1999, Molodstov[7] introduced the novel concept of soft sets and established the fundamental results of the new theory. In 2003, Maji *et al.*[5] studied some properties of soft sets. Pei and Miao [9] and Chen [1]*et al.,* improved the work of Maji *et al.* [4, 5].In [2], Jun *et al.*, introduced a new notion, of cubic set by using a fuzzy set and an interval-valued fuzzy set, and investigate several properties.

Muhiuddin and Al-roqi [8], introduced the notions of internal, external cubic soft sets, P-cubic (R-cubic) soft subsets, R-union(R-intersection, P-union and P-intersection) of cubic soft sets and the complement of a cubic soft set. They investigated several related properties and applied the notion of cubic soft sets to BCK/BCI-algebras.

Fuzzy matrix was introduced by Thomason [10] and the concept of uncertainty was discussed by using fuzzy matrices. Different concepts and ideas of fuzzy matrices have been given earlier mainly by Kim, Meenakshi and Thomson [3, 6, 10]. Fuzzy matrix plays a vital role in fuzzy modeling, fuzzy diagnosis and fuzzy controls. It also has applications in fields like psychology, medicine, economics and sociology.

In this paper, we introduce notion of cubic soft matrix. We defined internal cubic soft matrix, external cubic soft matrix, P-(R-)order, P-(R-)union, P-(R-) intersection, P-OR, R-OR, P-AND and R-AND of cubic soft matrix and their properties are discussed.

## 2. Preliminaries

In this section first we review some basic concepts and definitions.

*Corresponding author. Email: kv.chinnadurai@yahoo.com

Definition 2.1 *[8] Let $U$ be an initial universal set and $E$ be a set of parameters. A cubic soft set over $U$ is defined to be a pair $(\mathscr{F}, A)$ where $\mathscr{F}$ is a mapping from $A$ to $P(U)$ and $A \subseteq E$. Then the pair $(\mathscr{F}, A)$ can be represented as,*
$(\mathscr{F}, A) = \left\{ \mathscr{F}(e) \big/ e \in A \right\}$ *where* $\mathscr{F}(e) = \left\{ \left\langle u, \tilde{A}_e(u), \lambda_e(u) \right\rangle \big/ u \in U, e \in A \right\}$ *is a cubic soft set in which $\tilde{A}_e(u)$ is the interval valued fuzzy set and $\lambda_e(u)$ is a fuzzy set.*

Definition 2.2 *[8] Let $U$ be an initial universal set and $E$ be a set of parameters. A cubic soft set $(\mathscr{F}, A)$ over $U$ is said to be an internal cubic soft set if $A_e^-(u) \leqslant \lambda_e(u) \leqslant A_e^+(u)$ for all $e \in A$ and for all $u \in U$.*

Definition 2.3 *[8] Let $U$ be an initial universal set and $E$ be a set of parameters. A cubic soft set $(\mathscr{F}, A)$ over $U$ is said to be an external cubic soft set if $\lambda_e(u) \notin (A_e^-(u), A_e^+(u))$ for all $e \in A$ and for all $u \in U$.*

Definition 2.4 *[8] Let $U$ be an initial universal set and $E$ be a set of parameters. For any subsets $A$ and $B$ of $E$, $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be cubic soft sets over $U$.*

*(1) The R-union of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is a cubic soft set $(\mathscr{H}, C)$ where $C = A \cup B$ and*

$$\mathscr{H} e) = \begin{cases} \mathscr{F}(e) & if e \in A/B, \\ \mathscr{G}(e) & if e \in B/A, \\ \mathscr{F}(e) \cup_R \mathscr{G}(e) & if e \in A \cap B \end{cases}$$

*for all $e \in C$. This is denoted by $(\mathscr{H}, C) = (\mathscr{F}, A) \cup_R (\mathscr{G}, B)$.*

*(2) The R-intersection of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is a cubic soft set $(\mathscr{H}, C)$ where $C = A \cap B$ and*

$$\mathscr{H} e) = \begin{cases} \mathscr{F}(e) & if e \in A/B, \\ \mathscr{G}(e) & if e \in B/A, \\ \mathscr{F}(e) \cap_R \mathscr{G}(e) & if e \in A \cap B \end{cases}$$

*for all $e \in C$. This is denoted by $(\mathscr{H}, C) = (\mathscr{F}, A) \cap_R (\mathscr{G}, B)$.*

Definition 2.5 *[8] Let $U$ be an initial universal set and $E$ be a set of parameters. For any subsets $A$ and $B$ of $E$, $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be cubic soft sets over $U$.*

*(1) The P-union of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is a cubic soft set $(\mathscr{H}, C)$ where $C = A \cup B$ and*

$$\mathscr{H} e) = \begin{cases} \mathscr{F}(e) & if e \in A/B, \\ \mathscr{G}(e) & if e \in B/A, \\ \mathscr{F}(e) \cup_P \mathscr{G}(e) & if e \in A \cap B \end{cases}$$

*for all $e \in C$. This is denoted by $(\mathscr{H}, C) = (\mathscr{F}, A) \cup_P (\mathscr{G}, B)$.*

*(2) The P-intersection of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is a cubic soft set $(\mathscr{H}, C)$ where $C = A \cap B$ and*

$$\mathscr{H} e) = \begin{cases} \mathscr{F}(e) & if e \in A/B, \\ \mathscr{G}(e) & if e \in B/A, \\ \mathscr{F}(e) \cap_P \mathscr{G}(e) & if e \in A \cap B \end{cases}$$

*for all* $e \in C$. *This is denoted by* $(\mathscr{H}, C) = (\mathscr{F}, A) \cap_P (\mathscr{G}, B)$.

Definition 2.6 *[8] Let* $U$ *be an initial universal set and* $E$ *be a set of parameters. For any subsets* $A$ *and* $B$ *of* $E$, $(\mathscr{F}, A)$ *and* $(\mathscr{G}, B)$ *be cubic soft sets over* $U$.

(1) $R - OR$ *is denoted by* $(\mathscr{F}, A) \vee_R (\mathscr{G}, B)$ *and defined as* $(\mathscr{F}, A) \vee_R (\mathscr{G}, B) = (\mathscr{H}, A \times B)$ *where* $\mathscr{H}(\alpha, \beta) = \mathscr{F}(\alpha) \cup_R \mathscr{G}(\beta)$ *for all* $(\alpha, \beta) \in A \times B$.
(2) $R - AND$ *is denoted by* $(\mathscr{F}, A) \wedge_R (\mathscr{G}, B)$ *and defined as* $(\mathscr{F}, A) \wedge_R (\mathscr{G}, B) = (\mathscr{H}, A \times B)$ *where* $\mathscr{H}(\alpha, \beta) = \mathscr{F}(\alpha) \cap_R \mathscr{G}(\beta)$ *for all* $(\alpha, \beta) \in A \times B$.
(3) $P - OR$ *is denoted by* $(\mathscr{F}, A) \vee_P (\mathscr{G}, B)$ *and defined as* $(\mathscr{F}, A) \vee_P (\mathscr{G}, B) = (\mathscr{H}, A \times B)$ *where* $\mathscr{H}(\alpha, \beta) = \mathscr{F}(\alpha) \cup_P \mathscr{G}(\beta)$ *for all* $(\alpha, \beta) \in A \times B$.
(4) $P - AND$ *is denoted by* $(\mathscr{F}, A) \wedge_P (\mathscr{G}, B)$ *and defined as* $(\mathscr{F}, A) \wedge_P (\mathscr{G}, B) = (\mathscr{H}, A \times B)$ *where* $\mathscr{H}(\alpha, \beta) = \mathscr{F}(\alpha) \cap_P \mathscr{G}(\beta)$ *for all* $(\alpha, \beta) \in A \times B$.

Definition 2.7 *[8] Let* $U$ *be an initial universal set and* $E$ *be a set of parameters. The complement of a cubic soft set*$(\mathscr{F}, A)$ *over* $U$ *is denoted by* $(\mathscr{F}, A)^c$ *and defined by* $(\mathscr{F}, A) = (\mathscr{F}^c, \neg A)$ *where* $\mathscr{F}^c : \neg A \to \mathscr{C}P(U)$ *and* $(\mathscr{F}, A)^c = \left\{ \mathscr{F}^c(e) \big/ e \in A \right\}$ *where* $\mathscr{F}^c(e) = \left\{ \left\langle u, \tilde{A}_e^c(u), \lambda_e^c(u) \right\rangle \big/ u \in U, e \in A \right\}$.

Definition 2.8 *[6] A matrix* $A = [a_{ij}]_{m \times n}$ *is said to be fuzzy matrix if* $a_{ij} \in [0, 1], 1 \leqslant i \leqslant m$ *and* $1 \leqslant j \leqslant n$.

Definition 2.9 *[6] For any two fuzzy matrices* $A = [a_{ij}], B = [b_{ij}]$ *and a scalar* $k \in F$. *Then,*
*(i)* $A + B = [sup\{a_{ij}, b_{ij}\}] = \vee[a_{ij}, bij]$.
*(ii)* $AB = [sup\{inf\{a_{ij}, b_{ij}\}\}] = \vee\{\wedge[a_{ij}, b_{ij}]\}$.
*(iii)*$kA = [inf\{k, a_{ij}\}] = \wedge[k, a_{ij}]$.

Definition 2.10 *[6] For any matrix* $A = [a_{ij}]$, *the transpose is obtained by interchanging its rows and columns and is denoted by* $A^T = [a_{ji}]$ *for all i, j.*

## 3. Cubic Soft Matrices (CSM)

Definition 3.1 *Let* $U = \{u_1, u_2, ..., u_m\}$ *be an initial universal set and* $E = \{e_1, e_2, ..., e_n\}$ *be a set of parameters. Let* $A \subseteq E$. *Then cubic soft set* $(\mathscr{F}, A)$ *can be expressed in matrix form as*

$$A^{\boxplus} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

*such that* $A^{\boxplus} = [a_{ij}] = \left\langle \tilde{A}_{e_j}(u_i), \lambda_{e_j}(u_i) \right\rangle = \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle$ *which is called an* $m \times n$ *cubic soft matrix(shortly CS-matrix or CSM) of the cubic soft set* $(\mathscr{F}, A)$, *where* $i = 1, 2, 3, ..., m$ *and* $j = 1, 2, 3, ..., n$.

*Example 3.2* Let $U = \{u_1, u_2, u_3, u_4\}$ is a set of cars and $A = \{e_1, e_2, e_3\}$ is a set of parameters, which stands for mileage, engine and prize respectively. Then cubic soft set is defined as

$$(\mathscr{F}, A) = \Big\{ [e_1, (u_1, \langle [0.5, 0.8], 0.6\rangle), (u_2, \langle [0.1, 0.7], 0.5\rangle), (u_3, \langle [0.2, 0.6], 0.9\rangle), (u_4, \langle [0.3, 0.9], 0.4\rangle)] ,$$
$$[e_2, (u_1, \langle [0.2, 0.5], 0.3\rangle), (u_2, \langle [0.3, 0.6], 0.7\rangle), (u_3, \langle [0.2, 0.7], 0.2\rangle), (u_4, \langle [0.3, 0.5], 0.1\rangle)] ,$$
$$[e_3, (u_1, \langle [0.1, 0.8], 0.4\rangle), (u_2, \langle [0.6, 0.7], 0.9\rangle), (u_3, \langle [0.2, 0.9], 0.5\rangle), (u_4, \langle [0.3, 0.7], 0.4\rangle)] \Big\}.$$

Then the CS-matrix $A^{\boxplus}$ is written as,

$$A^{\boxplus} = \begin{bmatrix} \langle [0.5, 0.8], 0.6\rangle & \langle [0.2, 0.5], 0.3\rangle & \langle [0.1, 0.8], 0.4\rangle \\ \langle [0.1, 0.7], 0.5\rangle & \langle [0.3, 0.6], 0.7\rangle & \langle [0.6, 0.7], 0.9\rangle \\ \langle [0.2, 0.6], 0.7\rangle & \langle [0.2, 0.7], 0.2\rangle & \langle [0.2, 0.9], 0.5\rangle \\ \langle [0.3, 0.9], 0.4\rangle & \langle [0.3, 0.5], 0.1\rangle & \langle [0.3, 0.7], 0.4\rangle \end{bmatrix}.$$

DEFINITION 3.3 *A cubic soft matrix of order $1 \times n$ is called a row-cubic soft matrix. i.e., The universal set contains only one element.*

*Example 3.4* Let $U = \{u_1\}$ and $A = \{e_1, e_2, e_3\}$. Then$(\mathscr{F}, A)$ be a cubic soft set is defined as,
$(\mathscr{F}, A) = \{[e_1, (u_1, \langle [0.4, 0.8], 0.5\rangle)], [e_2, (u_1, \langle [0.5, 0.9], 0.7\rangle)], [e_3, (u_1, \langle [0.2, 0.6], 0.1\rangle)]\}$
Then, the CS-matrix $A^{\boxplus}$ is given by,

$$A^{\boxplus} = [\langle [0.4, 0.8], 0.5\rangle \ \langle [0.5, 0.9], 0.7\rangle \ \langle [0.2, 0.6], 0.1\rangle] .$$

DEFINITION 3.5 *A cubic soft matrix of order $m \times 1$ is called a column-cubic soft matrix. i.e., The parameter set contains only one parameter.*

*Example 3.6* Let $U = \{u_1, u_2, u_3, u_4\}$ and $A = \{e_1\}$.
Then $(\mathscr{F}, A)$ be a cubic soft set as,

$$(\mathscr{F}, A) = \Big\{ [e_1, (u_1, \langle [0.1, 0.4], 0.5\rangle), (u_2, \langle [0.3, 0.7], 1\rangle), (u_3, \langle [0.5, 0.8], 0.6\rangle), (u_4, \langle [0.4, 0.3], 0.3\rangle)] \Big\}$$

Then, the CS-matrix $A^{\boxplus}$ is given by,

$$A^{\boxplus} = \begin{bmatrix} \langle [0.1, 0.4], 0.5\rangle \\ \langle [0.3, 0.7], 1\rangle \\ \langle [0.5, 0.8], 0.6\rangle \\ \langle [0.4, 0.3], 0.3\rangle \end{bmatrix}.$$

DEFINITION 3.7 *A cubic soft matrix of order $m \times n$ is said to be a square-cubic soft matrix if $m = n$ i.e., the number of rows and the number of columns are equal.*

*Example 3.8* Let $U = \{u_1, u_2, u_3\}$ and $A = \{e_1, e_2, e_3\}$. Then $(\mathscr{F}, A)$ be a cubic soft set is defined as,

$$(\mathscr{F}, A) = \Big\{ [e_1, (u_1, \langle [0.7, 0.9], 0.6\rangle), (u_2, \langle [0.4, 0.8], 0.1\rangle), (u_3, \langle [0.1, 0.6], 0.6\rangle)] ,$$
$$[e_2, (u_1, \langle [0.2, 0.8], 0.6\rangle), (u_2, \langle [0.2, 0.7], 0.3\rangle), (u_3, \langle [0.1, 0.8], 0.5\rangle)] ,$$
$$[e_3, (u_1, \langle [0.2, 1], 0.4\rangle), (u_2, \langle [0.4, 0.8], 0.1\rangle), (u_3, \langle [0.1, 0.9], 0.2\rangle)] \Big\}$$

Then the CS-matrix $A^{\boxplus}$ is given by,

$$A^{\boxplus} = \begin{bmatrix} \langle[0.7,0.9],0.6\rangle & \langle[0.2,0.8],0.6\rangle & \langle[0.2,1],0.4\rangle \\ \langle[0.4,0.8],0.1\rangle & \langle[0.2,0.7],0.3\rangle & \langle[0.4,0.8],0.1\rangle \\ \langle[0.1,0.6],0.6\rangle & \langle[0.1,0.8],0.5\rangle & \langle[0.1,0.9],0.2\rangle \end{bmatrix}.$$

DEFINITION 3.9 *The transpose of a cubic soft matrix $A^{\boxplus}_{m\times n}$ is obtained by interchanging its rows and columns. It is denoted by $\left(A^{\boxplus}\right)^{T}_{n\times m}$.*

*Example 3.10* Consider the example 3.8, then its transpose cubic soft matrix as,

$$\left(A^{\boxplus}\right)^{T} = \begin{bmatrix} \langle[0.7,0.9],0.6\rangle & \langle[0.4,0.8],0.1\rangle & \langle[0.1,0.6],0.6\rangle \\ \langle[0.2,0.8],0.6\rangle & \langle[0.2,0.7],0.3\rangle & \langle[0.1,0.8],0.5\rangle \\ \langle[0.2,1],0.4\rangle & \langle[0.4,0.8],0.1\rangle & \langle[0.1,0.9],0.2\rangle \end{bmatrix}.$$

DEFINITION 3.11 *A square cubic soft matrix $A^{\boxplus}$ of order $n \times n$ is said to be a symmetric cubic soft matrix, if its transpose be equal to it, i.e., $\left(A^{\boxplus}\right)^{T} = A^{\boxplus}$.*

*Example 3.12* Let $U = \{u_1, u_2, u_3\}$ and $A = \{e_1, e_2, e_3\}$. Then $(\mathscr{F}, A)$ be a cubic soft set is defined as,

$$\begin{aligned}(\mathscr{F}, A) = \Big\{ & [e_1, (u_1, \langle[0.2,0.7],0.1\rangle), (u_2, \langle[0.3,0.5],0.9\rangle), (u_3, \langle[0.5,1],0.6\rangle)], \\ & [e_2, (u_1, \langle[0.3,0.5],0.9\rangle), (u_2, \langle[0.1,0.3],0.3\rangle), (u_3, \langle[0.5,0.9],0.5\rangle)], \\ & [e_3, (u_1, \langle[0.5,1],0.6\rangle), (u_2, \langle[0.5,0.9],0.5\rangle), (u_3, \langle[0.1,0.9],0.2\rangle)] \Big\}\end{aligned}$$

Then the symmetric matrix $A^{\boxplus}$ is given by,

$$A^{\boxplus} = \begin{bmatrix} \langle[0.2,0.7],0.1\rangle & \langle[0.3,0.5],0.9\rangle & \langle[0.5,1],0.6\rangle \\ \langle[0.3,0.5],0.9\rangle & \langle[0.1,0.3],0.3\rangle & \langle[0.5,0.9],0.5\rangle \\ \langle[0.5,1],0.6\rangle & \langle[0.5,0.9],0.5\rangle & \langle[0.1,0.9],0.2\rangle \end{bmatrix}$$

DEFINITION 3.13 *Let $A^{\boxplus} = [a_{ij}] \in CSM_{m\times n}$. Then $A^{\boxplus}$ is an internal cubic soft matrix (ICSM), if $\tilde{A}^{a^-}_{ij} \leqslant \lambda^{a}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$ for all i, j.*

*Example 3.14* Let $U = \{u_1, u_2, u_3, u_4\}$and $A = \{e_1, e_2, e_3\}$. Then $(\mathscr{F}, A)$ be a cubic soft set is defined as,

$$\begin{aligned}(\mathscr{F}, A) = \Big\{ & [e_1, (u_1, \langle[0.6,0.8],0.5\rangle), (u_2, \langle[0.4,0.9],0.7\rangle), (u_3, \langle[0.5,0.9],0.5\rangle), (u_4, \langle[0.3,0.7],0.5\rangle)], \\ & [e_2, (u_1, \langle[0.5,0.9],0.5\rangle), (u_2, \langle[0.3,0.8],0.6\rangle), (u_3, \langle[0.7,1],0.8\rangle), (u_4, \langle[0.6,0.8],0.75\rangle)], \\ & [e_3, (u_1, \langle[0.4,0.7],0.6\rangle), (u_2, \langle[0.3,0.8],0.5\rangle), (u_3, \langle[0.2,0.5],0.35\rangle), (u_4, \langle[0.2,0.5],0.3\rangle)] \Big\}\end{aligned}$$

Then the ICSM $A^{\boxplus}$ is given as,

$$A^{\boxplus} = \begin{bmatrix} \langle[0.6,0.8],0.5\rangle & \langle[0.5,0.9],0.5\rangle & \langle[0.4,0.7],0.6\rangle \\ \langle[0.4,0.9],0.7\rangle & \langle[0.3,0.8],0.6\rangle & \langle[0.3,0.8],0.5\rangle \\ \langle[0.5,0.9],0.5\rangle & \langle[0.7,1],0.8\rangle & \langle[0.2,0.5],0.35\rangle \\ \langle[0.3,0.7],0.5\rangle & \langle[0.6,0.8],0.75\rangle & \langle[0.2,0.5],0.3\rangle \end{bmatrix}.$$

DEFINITION 3.15 *Let $A^{\boxplus} = [a_{ij}] \in CSM_{m\times n}$. Then $A^{\boxplus}$ is an external cubic soft matrix (ECSM), if $\lambda^{a}_{ij} \notin (\tilde{A}^{a^-}_{ij}, \tilde{A}^{a^+}_{ij})$ for all i, j.*

*Example 3.16* Let $U = \{u_1, u_2, u_3, u_4\}$ and $E = \{e_1, e_2, e_3\}$ .Then $(\mathscr{F}, A)$ be a cubic soft set is defined as,

$$(\mathscr{F}, A) = \Big\{ [e_1, (u_1, \langle[0.3, 0.6], 0.8\rangle), (u_2, \langle[0.6, 1], 0.1\rangle), (u_3, \langle[0.4, 0.9], 0.2\rangle), (u_4, \langle[0.2, 0.6], 0.65\rangle)],$$
$$[e_2, (u_1, \langle[0.6, 0.9], 1\rangle), (u_2, \langle[0.3, 0.6], 0.75\rangle), (u_3, \langle[0.2, 0.7], 0.1\rangle), (u_4, \langle[0.6, 0.9], 0.4\rangle)],$$
$$[e_3, (u_1, \langle[0.2, 0.6], 0.6\rangle), (u_2, \langle[0.1, 0.9], 1\rangle), (u_3, \langle[0.4, 0.5], 0.7\rangle), (u_4, \langle[0.2, 0.6], 0.75\rangle)] \Big\}$$

Then the ECSM $A^{\boxplus}$ is given by,

$$A^{\boxplus} = \begin{bmatrix} \langle[0.3, 0.6], 0.8\rangle & \langle[0.6, 0.9], 1\rangle & \langle[0.2, 0.6], 0.6\rangle \\ \langle[0.6, 1], 0.1\rangle & \langle[0.3, 0.6], 0.75\rangle & \langle[0.1, 0.9], 1\rangle \\ \langle[0.4, 0.9], 0.2\rangle & \langle[0.2, 0.7], 0.1\rangle & \langle[0.4, 0.5], 0.7\rangle \\ \langle[0.2, 0.6], 0.65\rangle & \langle[0.6, 0.9], 0.4\rangle & \langle[0.2, 0.6], 0.75\rangle \end{bmatrix}.$$

DEFINITION 3.17 *Let $A = [a_{ij}] \in CSM_{m\times n}$. Then $[a_{ij}]$ is called*
*i) A zero CS-matrix, denoted by $\hat{0}$ if $[a_{ij}] = \langle[0, 0], 0\rangle$ for all i, j.*
*ii) A universal CS-matrix, denoted by $\hat{1}$ if $[a_{ij}] = \langle[1, 1], 1\rangle$ for all i, j.*

*Example 3.18* Let $U = \{u_1, u_2, u_3\}$ be the universal set and $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the set of parameters. Let $A, B \subseteq E$. Then

$$(\mathscr{F}, A) = \Big\{ [e_1, (u_1, \langle[0, 0], 0\rangle), (u_2, \langle[0, 0], 0\rangle), (u_3, \langle[0, 0], 0\rangle)],$$
$$[e_2, (u_1, \langle[0, 0], 0\rangle), (u_2, \langle[0, 0], 0\rangle), (u_3, \langle[0, 0], 0\rangle)],$$
$$[e_3, (u_1, \langle[0, 0], 0\rangle), (u_2, \langle[0, 0], 0\rangle), (u_3, \langle[0, 0], 0\rangle)] \Big\}$$

and

$$(\mathscr{F}, B) = \Big\{ [e_1, (u_1, \langle[1, 1], 1\rangle), (u_2, \langle[1, 1], 1\rangle), (u_3, \langle[1, 1], 1\rangle)],$$
$$[e_2, (u_1, \langle[1, 1], 1\rangle), (u_2, \langle[1, 1], 1\rangle), (u_3, \langle[1, 1], 1\rangle)],$$
$$[e_3, (u_1, \langle[1, 1], 1\rangle), (u_2, \langle[1, 1], 1\rangle), (u_3, \langle[1, 1], 1\rangle)] \Big\}.$$

A zero CS-matrix $A^{\boxplus}$ is given by,

$$A^{\boxplus} = \begin{bmatrix} \langle[0, 0], 0\rangle & \langle[0, 0], 0\rangle & \langle[0, 0], 0\rangle \\ \langle[0, 0], 0\rangle & \langle[0, 0], 0\rangle & \langle[0, 0], 0\rangle \\ \langle[0, 0], 0\rangle & \langle[0, 0], 0\rangle & \langle[0, 0], 0\rangle \end{bmatrix}.$$

A universal CS-matrix $A^{\boxplus}$ is given by,

$$B^{\boxplus} = \begin{bmatrix} \langle[1, 1], 1\rangle & \langle[1, 1], 1\rangle & \langle[1, 1], 1\rangle \\ \langle[1, 1], 1\rangle & \langle[1, 1], 1\rangle & \langle[1, 1], 1\rangle \\ \langle[1, 1], 1\rangle & \langle[1, 1], 1\rangle & \langle[1, 1], 1\rangle \end{bmatrix}$$

DEFINITION 3.19 *Let $A^{\boxplus} = [a_{ij}]_{m\times n}$, $B^{\boxplus} = [b_{ij}]_{m\times n}$ be the two cubic soft matrix of order $m \times n$. Then P-order matrix is denoted and defined as $[a_{ij}] \subseteq_P [b_{ij}]$, if $\tilde{A}^a_{ij} \leqslant \tilde{B}^b_{ij}$ and $\lambda^a_{ij} \leqslant \mu^b_{ij}$ for all i, j.*

*Example 3.20* Let $A^{\boxplus}$ and $B^{\boxplus}$ be the cubic soft matrices are defined as follows,

$$A^{\boxplus} = \begin{bmatrix} \langle[0.2,0.5],0.7\rangle & \langle[0.1,0.4],0.2\rangle & \langle[0.4,0.7],0.5\rangle \\ \langle[0.3,0.6],0.5\rangle & \langle[0.6,0.9],0.4\rangle & \langle[0.6,0.9],0.7\rangle \\ \langle[0.4,0.7],0.5\rangle & \langle[0.7,1],0.8\rangle & \langle[0.3,0.6],0.5\rangle \\ \langle[0.5,0.8],0.6\rangle & \langle[0.3,0.6],0.5\rangle & \langle[0.2,0.5],0.7\rangle \end{bmatrix}$$

and

$$B^{\boxplus} = \begin{bmatrix} \langle[0.3,0.6],0.8\rangle & \langle[0.2,0.5],0.3\rangle & \langle[0.5,0.8],0.6\rangle \\ \langle[0.4,0.7],0.6\rangle & \langle[0.6,0.9],0.5\rangle & \langle[0.7,0.9],0.8\rangle \\ \langle[0.5,0.8],0.5\rangle & \langle[0.7,1],0.8\rangle & \langle[0.4,0.7],0.6\rangle \\ \langle[0.6,0.9],0.7\rangle & \langle[0.4,0.7],0.6\rangle & \langle[0.3,0.6],0.7\rangle \end{bmatrix}.$$

Then $A^{\boxplus} \subseteq_P B^{\boxplus}$.

DEFINITION 3.21 *Let $A^{\boxplus} = [a_{ij}]_{m\times n}$, $B^{\boxplus} = [b_{ij}]_{m\times n}$ be the two cubic soft matrix of order $m \times n$. Then R-order matrix is denoted and defined as $[a_{ij}] \subseteq_R [b_{ij}]$, if $\tilde{A}^a_{ij} \leqslant \tilde{B}^b_{ij}$ and $\lambda^a_{ij} \geqslant \mu^b_{ij}$ for all i, j.*

*Example 3.22* Let $A^{\boxplus}$ and $B^{\boxplus}$ be the cubic soft matrices are defined as follows,

$$A^{\boxplus} = \begin{bmatrix} \langle[0.3,0.6],0.7\rangle & \langle[0.1,0.4],0.6\rangle & \langle[0.2,0.5],0.3\rangle \\ \langle[0.4,0.7],0.3\rangle & \langle[0.2,0.5],0.2\rangle & \langle[0.6,0.9],1\rangle \\ \langle[0.5,0.8],0.4\rangle & \langle[0.6,0.9],0.2\rangle & \langle[0.5,0.8],0.7\rangle \\ \langle[0.6,0.9],0.5\rangle & \langle[0.4,0.7],0.6\rangle & \langle[0.1,0.4],0.9\rangle \end{bmatrix}$$

and

$$B^{\boxplus} = \begin{bmatrix} \langle[0.4,0.7],0.6\rangle & \langle[0.2,0.5],0.5\rangle & \langle[0.3,0.6],0.2\rangle \\ \langle[0.5,0.8],0.2\rangle & \langle[0.3,0.6],0.1\rangle & \langle[0.6,0.9],0.6\rangle \\ \langle[0.6,0.9],0.3\rangle & \langle[0.7,0.9],0.2\rangle & \langle[0.6,0.9],0.5\rangle \\ \langle[0.7,0.9],0.4\rangle & \langle[0.5,0.8],0.5\rangle & \langle[0.4,0.7],0.9\rangle \end{bmatrix}.$$

Then $A^{\boxplus} \subseteq_R B^{\boxplus}$.

DEFINITION 3.23 *Let $A^{\boxplus} = [a_{ij}]_{m\times n}$, $B^{\boxplus} = [b_{ij}]_{m\times n}$ be the two cubic soft matrix of order $m \times n$. Then equal matrix is denoted and defined as $[a_{ij}] = [b_{ij}]$, if $\tilde{A}^{a^-}_{ij} = \tilde{B}^{b^-}_{ij}$ and $\lambda^a_{ij} = \mu^b_{ij}$ for all i, j.*

*Example 3.24* Let $A^{\boxplus}$ and $B^{\boxplus}$ be the cubic soft matrices is defined as follows,

$$A^{\boxplus} = \begin{bmatrix} \langle[0.5,0.8],0.2\rangle & \langle[0.3,0.6],0.1\rangle & \langle[0.6,0.9],0.6\rangle \\ \langle[0.6,0.9],0.3\rangle & \langle[0.7,0.9],0.2\rangle & \langle[0.6,0.9],0.5\rangle \\ \langle[0.7,0.9],0.4\rangle & \langle[0.5,0.8],0.5\rangle & \langle[0.4,0.7],0.9\rangle \end{bmatrix}$$

and

$$B^{\boxplus} = \begin{bmatrix} \langle[0.5,0.8],0.2\rangle & \langle[0.3,0.6],0.1\rangle & \langle[0.6,0.9],0.6\rangle \\ \langle[0.6,0.9],0.3\rangle & \langle[0.7,0.9],0.2\rangle & \langle[0.6,0.9],0.5\rangle \\ \langle[0.7,0.9],0.4\rangle & \langle[0.5,0.8],0.5\rangle & \langle[0.4,0.7],0.9\rangle \end{bmatrix}.$$

Then $A^{\boxplus} = B^{\boxplus}$.

## 4. P-union, P-intersection, R-union and R-intersection of cubic soft matrices

In this section we have defined P-union, P-intersection, R-union and R-intersection of any two cubic soft matrices and investigate some of its properties.

DEFINITION 4.1 *Let* $A^{\boxplus} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^{a} \right\rangle, B^{\boxplus} = \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^{b} \right\rangle$ $\in CSM_{m\times n}$. *Then*

(1) *P-union of* $A^{\boxplus}$ *and* $B^{\boxplus}$ *is denoted by* $A^{\boxplus} \vee_P B^{\boxplus}$ *and defined as* $A^{\boxplus} \vee_P B^{\boxplus} = C^{\boxplus}$, *if* $C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle$, *where* $\tilde{C}_{ij}^{c} = \max\left\{ \tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b} \right\}$ *and* $\gamma_{ij}^{c} = \max\left\{ \lambda_{ij}^{a}, \mu_{ij}^{b} \right\}$ *for all i, j.*

(2) *P-intersection of* $A^{\boxplus}$ *and* $B^{\boxplus}$ *is denoted by* $A^{\boxplus} \wedge_P B^{\boxplus}$ *and defined as* $A^{\boxplus} \wedge_P B^{\boxplus} = C^{\boxplus}$, *if* $C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle$, *where* $\tilde{C}_{ij}^{c} = \min\left\{ \tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b} \right\}$ *and* $\gamma_{ij}^{c} = \min\left\{ \lambda_{ij}^{a}, \mu_{ij}^{b} \right\}$ *for all i, j.*

(3) *R-union of* $A^{\boxplus}$ *and* $B^{\boxplus}$ *is denoted by* $A^{\boxplus} \vee_R B^{\boxplus}$ *and defined as* $A^{\boxplus} \vee_R B^{\boxplus} = C^{\boxplus}$, *if* $C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle$, *where* $\tilde{C}_{ij}^{c} = \max\left\{ \tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b} \right\}$ *and* $\gamma_{ij}^{c} = \min\left\{ \lambda_{ij}^{a}, \mu_{ij}^{b} \right\}$ *for all i, j.*

(4) *R-intersection of* $A^{\boxplus}$ *and* $B^{\boxplus}$ *is denoted by* $A^{\boxplus} \wedge_R B^{\boxplus}$ *and defined as* $A^{\boxplus} \wedge_R B^{\boxplus} = C^{\boxplus}$, *if* $C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle$, *where* $\tilde{C}_{ij}^{c} = \min\left\{ \tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b} \right\}$ *and* $\gamma_{ij}^{c} = \max\left\{ \lambda_{ij}^{a}, \mu_{ij}^{b} \right\}$ *for all i, j.*

*Example 4.2* Let $A^{\boxplus}$ and $B^{\boxplus}$ be two cubic soft matrices of order $4 \times 3$ and defined as follows.

$$A^{\boxplus} = \begin{bmatrix} \langle [0.3, 0.6], 0.6 \rangle & \langle [0.1, 0.4], 0.5 \rangle & \langle [0.5, 0.8], 0.7 \rangle \\ \langle [0.4, 0.7], 0.7 \rangle & \langle [0.7, 1], 0.4 \rangle & \langle [0.6, 0.9], 0.5 \rangle \\ \langle [0.6, 0.9], 0.8 \rangle & \langle [0.4, 0.7], 0.6 \rangle & \langle [0.4, 0.7], 0.6 \rangle \\ \langle [0.2, 0.5], 0.3 \rangle & \langle [0.5, 0.8], 0.7 \rangle & \langle [0.3, 0.6], 0.8 \rangle \end{bmatrix}$$

and

$$B^{\boxplus} = \begin{bmatrix} \langle [0.4, 0.5], 0.7 \rangle & \langle [0.4, 0.6], 0.6 \rangle & \langle [0.6, 0.9], 0.1 \rangle \\ \langle [0.5, 0.6], 0.5 \rangle & \langle [0.5, 0.7], 0.7 \rangle & \langle [0.7, 1], 0.4 \rangle \\ \langle [0.7, 0.8], 0.7 \rangle & \langle [0.7, 1], 0.5 \rangle & \langle [0.5, 0.8], 0.3 \rangle \\ \langle [0.3, 0.4], 0.6 \rangle & \langle [0.4, 0.7], 0.8 \rangle & \langle [0.4, 0.7], 0.9 \rangle \end{bmatrix}.$$

(1) **(P-union)** $A^{\boxplus} \vee_P B^{\boxplus} = C^{\boxplus}$ is defined as,

$$C^{\boxplus} = \begin{bmatrix} \langle [0.4, 0.6], 0.7 \rangle & \langle [0.4, 0.6], 0.6 \rangle & \langle [0.6, 0.9], 0.7 \rangle \\ \langle [0.5, 0.7], 0.7 \rangle & \langle [0.7, 1], 0.7 \rangle & \langle [0.7, 1], 0.5 \rangle \\ \langle [0.7, 0.9], 0.8 \rangle & \langle [0.7, 1], 0.6 \rangle & \langle [0.5, 0.8], 0.6 \rangle \\ \langle [0.3, 0.5], 0.6 \rangle & \langle [0.5, 0.8], 0.8 \rangle & \langle [0.4, 0.7], 0.9 \rangle \end{bmatrix}.$$

(2) **(P-intersection)** $A^{\boxplus} \wedge_P B^{\boxplus} = C^{\boxplus}$ is defined as,

$$C^{\boxplus} = \begin{bmatrix} \langle [0.3, 0.5], 0.6 \rangle & \langle [0.1, 0.4], 0.5 \rangle & \langle [0.5, 0.8], 0.1 \rangle \\ \langle [0.4, 0.6], 0.5 \rangle & \langle [0.5, 0.7], 0.4 \rangle & \langle [0.6, 0.9], 0.4 \rangle \\ \langle [0.6, 0.8], 0.7 \rangle & \langle [0.4, 0.7], 0.5 \rangle & \langle [0.4, 0.7], 0.3 \rangle \\ \langle [0.2, 0.4], 0.3 \rangle & \langle [0.4, 0.7], 0.7 \rangle & \langle [0.3, 0.6], 0.8 \rangle \end{bmatrix}.$$

(3) **(R-union)** $A^{\boxplus} \vee_R B^{\boxplus} = C^{\boxplus}$ is defined as,

$$C^{\boxplus} = \begin{bmatrix} \langle[0.4,0.6],0.6\rangle & \langle[0.4,0.6],0.5\rangle & \langle[0.6,0.9],0.1\rangle \\ \langle[0.5,0.7],0.5\rangle & \langle[0.7,1],0.4\rangle & \langle[0.7,1],0.4\rangle \\ \langle[0.7,0.9],0.7\rangle & \langle[0.7,1],0.5\rangle & \langle[0.5,0.8],0.3\rangle \\ \langle[0.3,0.5],0.3\rangle & \langle[0.5,0.8],0.7\rangle & \langle[0.4,0.7],0.8\rangle \end{bmatrix}.$$

(4) **(R-intersection)** $A^{\boxplus} \wedge_R B^{\boxplus} = C^{\boxplus}$ is defined as,

$$C^{\boxplus} = \begin{bmatrix} \langle[0.3,0.5],0.7\rangle & \langle[0.1,0.4],0.6\rangle & \langle[0.5,0.8],0.7\rangle \\ \langle[0.4,0.6],0.7\rangle & \langle[0.5,0.7],0.7\rangle & \langle[0.6,0.9],0.5\rangle \\ \langle[0.6,0.8],0.8\rangle & \langle[0.4,0.7],0.6\rangle & \langle[0.4,0.7],0.6\rangle \\ \langle[0.2,0.4],0.6\rangle & \langle[0.4,0.7],0.8\rangle & \langle[0.3,0.6],0.9\rangle \end{bmatrix}.$$

PROPOSITION 4.3 *Let $A^{\boxplus}, B^{\boxplus}, C^{\boxplus}, D^{\boxplus} \in CSM_{m\times n}$. Then the following hold.*

*(1) If $A^{\boxplus} \subseteq_P B^{\boxplus}$ and $B^{\boxplus} \subseteq_P C^{\boxplus}$, then $A^{\boxplus} \subseteq_P C^{\boxplus}$.*
*(2) If $A^{\boxplus} \subseteq_P B^{\boxplus}$ and $A^{\boxplus} \subseteq_P C^{\boxplus}$, then $A^{\boxplus} \subseteq_P (B^{\boxplus} \wedge_P C^{\boxplus})$.*
*(3) If $A^{\boxplus} \subseteq_P B^{\boxplus}$ and $C^{\boxplus} \subseteq_P B^{\boxplus}$, then $(A^{\boxplus} \vee_P C^{\boxplus} \subseteq_P B^{\boxplus})$.*
*(4) If $A^{\boxplus} \subseteq_P B^{\boxplus}$ and $C^{\boxplus} \subseteq_P D^{\boxplus}$, then*
*i) $A^{\boxplus} \vee_P C^{\boxplus} \subseteq_P B^{\boxplus} \vee_P D^{\boxplus}$.*
*ii) $A^{\boxplus} \wedge_P C^{\boxplus} \subseteq_P B^{\boxplus} \wedge_P D^{\boxplus}$.*
*(5) If $A^{\boxplus} \subseteq_R B^{\boxplus}$ and $B^{\boxplus} \subseteq_R C^{\boxplus}$, then $A^{\boxplus} \subseteq_R C^{\boxplus}$.*
*(6) If $A^{\boxplus} \subseteq_R B^{\boxplus}$ and $A^{\boxplus} \subseteq_R C^{\boxplus}$, then $A^{\boxplus} \subseteq_R (B^{\boxplus} \wedge_R C^{\boxplus})$.*
*(7) If $A^{\boxplus} \subseteq_R B^{\boxplus}$ and $C^{\boxplus} \subseteq_R B^{\boxplus}$, then $(A^{\boxplus} \vee_R C^{\boxplus}) \subseteq_R B^{\boxplus}$.*
*(8) If $A^{\boxplus} \subseteq_R B^{\boxplus}$ and $C^{\boxplus} \subseteq_R D^{\boxplus}$, then*
*i) $A^{\boxplus} \vee_R C^{\boxplus} \subseteq_R B^{\boxplus} \vee_R D^{\boxplus}$.*
*ii) $A^{\boxplus} \wedge_R C^{\boxplus} \subseteq_R B^{\boxplus} \wedge_R D^{\boxplus}$.*

*Proof* Straightforward. ■

DEFINITION 4.4 *Let $A^{\boxplus} = [a_{ij}] \in CSM_{m\times n}$. Then the complement of the cubic soft matrix is denoted by $(A^{\boxplus})^c = [b_{ij}]$, if $[b_{ij}] = \left\langle [1 - \tilde{A}^{a^+}_{ij}, 1 - \tilde{A}^{a^-}_{ij}], 1 - \lambda^a_{ij} \right\rangle$ for all i, j.*

*Example 4.5* Let

$$A^{\boxplus} = \begin{bmatrix} \langle[0.3,0.5],0.6\rangle & \langle[0.1,0.4],0.5\rangle & \langle[0.5,0.8],0.7\rangle \\ \langle[0.4,0.6],0.5\rangle & \langle[0.5,0.7],0.4\rangle & \langle[0.6,0.9],0.4\rangle \\ \langle[0.6,0.8],0.7\rangle & \langle[0.4,0.7],0.5\rangle & \langle[0.4,0.7],0.3\rangle \\ \langle[0.2,0.4],0.3\rangle & \langle[0.4,0.7],0.7\rangle & \langle[0.3,0.6],0.8\rangle \end{bmatrix}$$

be a cubic soft matrix of order $4 \times 3$. Then the complement is defined as,

$$(A^{\boxplus})^c = \begin{bmatrix} \langle[0.5,0.7],0.4\rangle & \langle[0.6,0.9],0.5\rangle & \langle[0.2,0.5],0.3\rangle \\ \langle[0.4,0.6],0.5\rangle & \langle[0.3,0.5],0.6\rangle & \langle[0.1,0.4],0.6\rangle \\ \langle[0.2,0.4],0.3\rangle & \langle[0.3,0.6],0.5\rangle & \langle[0.3,0.6],0.7\rangle \\ \langle[0.6,0.8],0.7\rangle & \langle[0.3,0.6],0.3\rangle & \langle[0.4,0.7],0.2\rangle \end{bmatrix}.$$

THEOREM 4.6 *Let $A^{\boxplus} = [a_{ij}] \in ICSM_{m\times n}$. Then $(A^{\boxplus})^c \in ICSM_{m\times n}$.*

*Proof* Let $A^{\boxplus} = [a_{ij}] \in ICSM_{m\times n}$. This implies that,

$$\tilde{A}^{a^-}_{ij} \leqslant \lambda^a_{ij} \leqslant \tilde{A}^{a^+}_{ij} \text{for all } i, j.$$

Then $1 - \tilde{A}^{a^+}_{ij} \geqslant 1 - \lambda^a_{ij} \geqslant 1 - \tilde{A}^{a^-}_{ij}$ for all i, j.

$1-\tilde{A}^{a^-}_{ij} \leqslant 1-\lambda^a_{ij} \leqslant 1-\tilde{A}^{a^+}_{ij}$ for all i, j.
Thus $(A^{\boxplus})^c \in ICSM_{m\times n}$. ■

THEOREM 4.7 *Let* $A^{\boxplus} = [a_{ij}] \in ECSM_{m\times n}$. *Then* $(A^{\boxplus})^c \in ECSM_{m\times n}$.

*Proof* Let $A^{\boxplus} = [a_{ij}] \in ECSM_{m\times n}$. This implies that, $\lambda^a_{ij} \notin (\tilde{A}^{a^-}_{ij}, \tilde{A}^{a^+}_{ij})$ for all i, j. Since $\lambda^a_{ij} \notin (\tilde{A}^{a^-}_{ij}, \tilde{A}^{a^+}_{ij})$ and $0 \leqslant \tilde{A}^{a^-}_{ij} \leqslant \tilde{A}^{a^+}_{ij} \leqslant 1$. So we have,

$$\lambda^a_{ij} \leqslant \tilde{A}^{a^-}_{ij}$$

or

$$\tilde{A}^{a^+}_{ij} \leqslant \lambda^a_{ij}.$$

This implies, $1-\lambda^a_{ij} \geqslant 1-\tilde{A}^{a^-}_{ij}$ or $1-\tilde{A}^{a^+}_{ij} \geqslant 1-\lambda^a_{ij}$.
Thus $1-\lambda^a_{ij} \notin \left((1-\tilde{A}^{a^-}_{ij}), (1-\tilde{A}^{a^+}_{ij})\right)$ for all i, j.
Therefore $(A^{\boxplus})^c \in ECSM_{m\times n}$. ■

THEOREM 4.8 *Let* $A^{\boxplus}, B^{\boxplus} \in ICSM_{m\times n}$. *Then*
*(1)* $A^{\boxplus} \vee_P B^{\boxplus} \in ICSM_{m\times n}$.
*(2)* $A^{\boxplus} \wedge_P B^{\boxplus} \in ICSM_{m\times n}$.

*Proof* (1) Let $A^{\boxplus}$ and $B^{\boxplus} \in ICSM_{m\times n}$.
For

$$A^{\boxplus} = [a_{ij}], \text{ we have } \tilde{A}^{a^-}_{ij} \leqslant \lambda^a_{ij} \leqslant \tilde{A}^{a^+}_{ij} \text{ for all i j}$$
$$B^{\boxplus} = [b_{ij}], \text{ we have } \tilde{B}^{b^-}_{ij} \leqslant \mu^b_{ij} \leqslant \tilde{B}^{b^+}_{ij} \text{ for all i j.}$$

Then, $\max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\} \leqslant \max\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant \max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^+}_{ij}\}$ for all i, j. Now by Definition 4.1, we have $A^{\boxplus} \vee_P B^{\boxplus} = C^{\boxplus} = c_{ij} = \left\langle C^c_{ij}, \gamma^c_{ij}\right\rangle$, where $\tilde{C}^c_{ij} = \max\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}$ and $\gamma^c_{ij} = \max\{\lambda^a_{ij}, \mu^b_{ij}\}$ for all i, j.
Hence $A^{\boxplus} \vee_P B^{\boxplus} \in ICSM_{m\times n}$. (2) Let $A^{\boxplus} \wedge_P B^{\boxplus} = C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}^c_{ij}, \gamma^c_{ij}\right\rangle$, where $\tilde{C}^c_{ij} = \min\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}$ and $\gamma^c_{ij} = \min\{\lambda^a_{ij}, \mu^b_{ij}\}$ for all i, j. Also given that $A^{\boxplus}, B^{\boxplus} \in ICSM_{m\times n}$.
For

$$A^{\boxplus} = [a_{ij}], \text{ we have } \tilde{A}^{a^-}_{ij} \leqslant \lambda^a_{ij} \leqslant \tilde{A}^{a^+}_{ij} \text{ for all i j}$$
$$B^{\boxplus} = [b_{ij}], \text{ we have } \tilde{B}^{b^-}_{ij} \leqslant \mu^b_{ij} \leqslant \tilde{B}^{b^+}_{ij} \text{ for all i j.}$$

This implies, $\min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\} \leqslant \min\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant \min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^+}_{ij}\}$ for all i, j. Hence $C^{\boxplus} = A^{\boxplus} \wedge_P B^{\boxplus} \in ICSM_{m\times n}$. ■

The following example shows that the P-union and P-intersection of ECSMs need not be an ECSM.

*Example 4.9* Let $A^{\boxplus} = \begin{bmatrix} \langle[0.3, 0.7], 0.1\rangle & \langle[0.1, 0.5], 0.7\rangle \\ \langle[0.4, 0.8], 0.2\rangle & \langle[0.4, 1], 0.3\rangle \end{bmatrix}$ and
$B^{\boxplus} = \begin{bmatrix} \langle[0.7, 0.9], 0.4\rangle & \langle[0.6, 0.8], 0.3\rangle \\ \langle[0.3, 0.5], 0.8\rangle & \langle[0.2, 0.4], 0.7\rangle \end{bmatrix}$. Then

$$(1) A^{\boxplus} \vee_P B^{\boxplus} = \begin{bmatrix} \langle[0.7, 0.9], 0.4\rangle & \langle[0.6, 0.8], 0.7\rangle \\ \langle[0.4, 0.8], 0.8\rangle & \langle[0.4, 1], 0.7\rangle \end{bmatrix} \text{ is not an ECSM.}$$

$$(2) A^{\boxplus} \wedge_P B^{\boxplus} = \begin{bmatrix} \langle[0.3, 0.7], 0.1\rangle & \langle[0.1, 0.5], 0.3\rangle \\ \langle[0.3, 0.5], 0.2\rangle & \langle[0.2, 0.4], 0.3\rangle \end{bmatrix} \text{ is not an ECSM.}$$

The following example shows that the R-union and R-intersection of ICSMs need not be an ICSM.

*Example 4.10* Let $A^{\boxplus}= \begin{bmatrix} \langle[0.2, 0.9], 0.4\rangle & \langle[0.6, 0.9], 0.7\rangle \\ \langle[0.1, 0.5], 0.2\rangle & \langle[0.4, 1], 0.6\rangle \end{bmatrix}$ and
$B^{\boxplus}= \begin{bmatrix} \langle[0.7, 1], 0.8\rangle & \langle[0.1, 0.8], 0.5\rangle \\ \langle[0.4, 0.9], 0.6\rangle & \langle[0.2, 0.7], 0.4\rangle \end{bmatrix}$. Then

$$(1) A^{\boxplus} \vee_R B^{\boxplus} = \begin{bmatrix} \langle[0.7, 1], 0.4\rangle & \langle[0.6, 0.9], 0.5\rangle \\ \langle[0.4, 0.9], 0.2\rangle & \langle[0.4, 1], 0.4\rangle \end{bmatrix} \text{ is not an ICSM.}$$

$$(2) A^{\boxplus} \wedge_R B^{\boxplus} = \begin{bmatrix} \langle[0.2, 0.9], 0.8\rangle & \langle[0.1, 0.8], 0.7\rangle \\ \langle[0.1, 0.5], 0.6\rangle & \langle[0.2, 0.7], 0.6\rangle \end{bmatrix} \text{ is not an ICSM.}$$

The following example shows that the R-union and R-intersection of ECSMs need not be an ECSM.

*Example 4.11* (1) Let $A^{\boxplus}= \begin{bmatrix} \langle[0.2, 0.4], 0.7\rangle & \langle[0.4, 0.6], 0.8\rangle \\ \langle[0.3, 0.7], 0.9\rangle & \langle[0.3, 0.8], 0.2\rangle \end{bmatrix}$ and
$B^{\boxplus}= \begin{bmatrix} \langle[0.6, 0.8], 0.9\rangle & \langle[0.1, 0.4], 0.6\rangle \\ \langle[0.1, 0.4], 0.5\rangle & \langle[0.2, 0.7], 1\rangle \end{bmatrix}$.

$$\text{Then } A^{\boxplus} \vee_R B^{\boxplus} = \begin{bmatrix} \langle[0.6, 0.8], 0.7\rangle & \langle[0.4, 0.6], 0.6\rangle \\ \langle[0.3, 0.7], 0.5\rangle & \langle[0.3, 0.8], 0.2\rangle \end{bmatrix} \text{ is not an ECSM.}$$

(2) Let $A^{\boxplus}= \begin{bmatrix} \langle[0.2, 0.4], 0.1\rangle & \langle[0.4, 0.6], 0.1\rangle \\ \langle[0.3, 0.6], 0.7\rangle & \langle[0.2, 0.7], 0.1\rangle \end{bmatrix}$ and
$B^{\boxplus}= \begin{bmatrix} \langle[0.4, 0.8], 0.3\rangle & \langle[0.5, 0.9], 0.4\rangle \\ \langle[0.2, 0.8], 0.1\rangle & \langle[0.3, 0.6], 0.2\rangle \end{bmatrix}$.

$$\text{Then } A^{\boxplus} \wedge_R B^{\boxplus} = \begin{bmatrix} \langle[0.2, 0.4], 0.3\rangle & \langle[0.4, 0.6], 0.4\rangle \\ \langle[0.2, 0.6], 0.7\rangle & \langle[0.2, 0.6], 0.2\rangle \end{bmatrix} \text{ is not an ECSM.}$$

THEOREM 4.12 *Let $A^{\boxplus}, B^{\boxplus} \in ICSM_{m\times n}$, such that*

$$\max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\} \leqslant \min\{\lambda^a_{ij}, \mu^b_{ij}\}$$

*for all i, j. Then, the R-union of $A^{\boxplus}$ and $B^{\boxplus}$ is also an ICSM.*

*Proof* Let $A^{\boxplus}, B^{\boxplus} \in ICSM_{m\times n}$.
For

$$A^{\boxplus}, \text{ we have } \tilde{A}^{a^-}_{ij} \leqslant \lambda^a_{ij} \leqslant \tilde{A}^{a^+}_{ij} \text{ for all i, j.}$$
$$B^{\boxplus}, \text{ we have } \tilde{B}^{b^-}_{ij} \leqslant \mu^b_{ij} \leqslant \tilde{B}^{b^+}_{ij} \text{ for all i,j.}$$

Then, $\min\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant \max\{\tilde{A}^{a^+}_{ij} \tilde{B}^{b^+}_{ij}\}$.
Also given that $\max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\} \leqslant \min\{\lambda^a_{ij}, \mu^b_{ij}\}$ for all i, j.
It follows that,

$$\max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\} \leqslant \min\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant \max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^+}_{ij}\} \text{ for all } i, j.$$

Thus $A^{\boxplus} \vee_R B^{\boxplus} = C^{\boxplus}$ is an ICSM if $\max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\} \leqslant \min\{\lambda^a_{ij}, \mu^b_{ij}\}$ for all i, j.■

THEOREM 4.13 *Let $A^{\boxplus}, B^{\boxplus} \in ICSM_{m\times n}$, satisfying the following inequality*

$$\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^+}_{ij}\} \geqslant \max\{\lambda^a_{ij}, \mu^b_{ij}\}$$

*for all i, j. Then $A^{\boxplus} \wedge_R B^{\boxplus} \in ICSM_{m\times n}$.*

*Proof* Let $A^{\boxplus}, B^{\boxplus} \in ICSM_{m\times n}$. Then by Definition 3.13,

$$\tilde{A}_{ij}^{a^-} \leqslant \lambda_{ij}^a \leqslant \tilde{A}_{ij}^{a^+} \text{ and } \tilde{B}_{ij}^{b^-} \leqslant \mu_{ij}^b \leqslant \tilde{B}_{ij}^{b^+}.$$

This implies that $\min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\} \leqslant \max\{\lambda_{ij}^a, \mu_{ij}^b\}$.
Also since $A^{\boxplus} \wedge_R B^{\boxplus} = C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle$, where $\tilde{C}_{ij}^c = \min\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}$ and $\gamma_{ij}^c = \max\{\lambda_{ij}^a, \mu_{ij}^b\}$ for all i, j. Then, $\min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\} \geqslant \max\{\lambda_{ij}^a, \mu_{ij}^b\}$.
Again by the Definition 3.13,

$$\min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\} \leqslant \max\{\lambda_{ij}^a, \mu_{ij}^b\} \leqslant \min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}.$$

Hence$A^{\boxplus} \wedge_R B^{\boxplus} \in ICSM_{m\times n}$. ■

Theorem 4.14 *Let $A^{\boxplus}, B^{\boxplus} \in ICSM_{m\times n}$, such that*

$$\min\{\lambda_{ij}^a, \mu_{ij}^a\} \leqslant max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}$$

*for all i, j. Then $A^{\boxplus} \vee_R B^{\boxplus} \in ECSM_{m\times n}$.*

*Proof* Let $A^{\boxplus}, B^{\boxplus} \in ICSM_{m\times n}$. By the Definition 3.13,

$$\tilde{A}_{ij}^{a^-} \leqslant \lambda_{ij}^a \leqslant \tilde{A}_{ij}^{a^+} \text{ and } \tilde{B}_{ij}^{b^-} \leqslant \mu_{ij}^b \leqslant \tilde{B}_{ij}^{b^+}.$$

Since $A^{\boxplus} \vee_R B^{\boxplus} = C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle$, where $\tilde{C}_{ij}^c = \max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}$ and $\gamma_{ij}^c = \min\{\lambda_{ij}^a, \mu_{ij}^b\}$ for all i, j. By hypothesis,

$$\min\{\lambda_{ij}^a, \mu_{ij}^b\} \leqslant \max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}.$$

This implies that
$\min\{\lambda_{ij}^a, \mu_{ij}^b\} \notin \left((\tilde{A}_{ij}^a \cup \tilde{B}_{ij}^b)^-, (\tilde{A}_{ij}^a \cup \tilde{B}_{ij}^b)^+\right) = (\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\})$.
Hence $A^{\boxplus} \vee_R B^{\boxplus}$ is an $ECSM$. ■

Theorem 4.15 *Let $A^{\boxplus}, B^{\boxplus} \in ICSM_{m\times n}$, such that*

$$\max\{\lambda_{ij}^a, \mu_{ij}^a\} \geqslant max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}$$

*for all i, j. Then $A^{\boxplus} \wedge_R B^{\boxplus}$ is an $ECSM_{m\times n}$.*

*Proof* Let $A^{\boxplus}, B^{\boxplus} \in ICSM_{m\times n}$. Then by the Definition 3.13,

$$\tilde{A}_{ij}^{a^-} \leqslant \lambda_{ij}^a \leqslant \tilde{A}_{ij}^{a^+} \text{ and } \tilde{B}_{ij}^{b^-} \leqslant \mu_{ij}^b \leqslant \tilde{B}_{ij}^{b^+}.$$

Since $A^{\boxplus} \wedge_R B^{\boxplus} = C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle$, where $\tilde{C}_{ij}^c = \min\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}$ and $\gamma_{ij}^c = \max\{\lambda_{ij}^a, \mu_{ij}^b\}$ for all i, j. Thus $\max\{\lambda_{ij}^a, \mu_{ij}^b\} \geqslant \min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}$.
This implies that
$\max\{\lambda_{ij}^a, \mu_{ij}^b\} \notin \left((\tilde{A}_{ij}^a \cup \tilde{B}_{ij}^b)^-, (\tilde{A}_{ij}^a \cup \tilde{B}_{ij}^b)^+\right) = (\min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\})$.
Hence $A^{\boxplus} \wedge_R B^{\boxplus}$ is an $ECSM$. ■

Theorem 4.16 *Let $A^{\boxplus}, B^{\boxplus} \in CSM_{m\times n}$, such that*
$\min\left\{\max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\} = \min\{\lambda_{ij}^a, \mu_{ij}^b\} =$
$\max\left\{\min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\}$ *for all i, j. Then, $A^{\boxplus} \wedge_P B^{\boxplus}$ is both ECSM and ICSM.*

*Proof* Consider $A^{\boxplus} \wedge_P B^{\boxplus} = C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle$, where $\tilde{C}_{ij}^c = \min\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}$ and $\gamma_{ij}^c = \min\{\lambda_{ij}^a, \mu_{ij}^b\}$. For each $e_j \in E$, take

$$\alpha_j = \min\left\{\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}$$
$$\beta_j = \max\left\{\min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}.$$

Then $\alpha_j$ is one of $\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}, \tilde{A}^{a^+}_{ij}$ and $\tilde{B}^{b^+}_{ij}$. We consider $\alpha_j = \tilde{A}^{a^-}_{ij}$ or $\tilde{A}^{a^+}_{ij}$ only, as the remaining cases are similar to this one. If $\alpha_j = \tilde{A}^{a^-}_{ij}$, then $\tilde{B}^{b^-}_{ij} \leqslant \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^-}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$ and if $\beta_j = \tilde{B}^{b^+}_{ij}$, then $\tilde{A}^{a^-}_{ij} = \alpha_j = \min\{\lambda^a_{ij}, \mu^b_{ij}\} = \beta_j = \tilde{B}^{b^+}_{ij}$. Thus

$$\tilde{B}^{b^-}_{ij} \leqslant \tilde{B}^{b^+}_{ij} = \min\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{A}^{a^-}_{ij} \leqslant \tilde{A}^{a^+}_{ij},$$

which implies that $\min\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{B}^{b^+}_{ij} = (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+$. Hence

$$\min\{\lambda^a_{ij}, \mu^b_{ij}\} \notin \left((\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+\right) \text{ and}$$
$$(\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^- \leqslant \min\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+.$$

If $\alpha_j = \tilde{A}^{a^+}_{ij}$, then $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$ and so $\min\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{A}^{a^+}_{ij} = (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+$. Hence

$$\min\{\lambda^a_{ij}, \mu^b_{ij}\} \notin \left((\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+\right) \text{ and}$$
$$(\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^- \leqslant \min\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+.$$

Consequently, we note that $A^{\boxplus} \wedge_P B^{\boxplus}$ is both ECSM and ICSM. ■

THEOREM 4.17 *Let $A^{\boxplus}, B^{\boxplus} \in CSM_{m\times n}$, such that* $\min\left\{\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\} = \min\{\lambda^a_{ij}, \mu^b_{ij}\} =$ $\max\left\{\min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}$ *for all i, j. Then, $A^{\boxplus} \wedge_R B^{\boxplus}$ is both ECSM and ICSM.*

*Proof* Consider $A^{\boxplus} \wedge_R B^{\boxplus} = C^{\boxplus}$, $C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}^c_{ij}, \gamma^c_{ij} \right\rangle$, where $\tilde{C}^c_{ij} = \min\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}$and $\gamma^c_{ij} = \max\{\lambda^a_{ij}, \mu^b_{ij}\}$. For each $e_j \in E$, take

$$\alpha_j = \min\left\{\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}, \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}$$
$$\beta_j = \max\left\{\min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}.$$

Then $\alpha_j$ is one of $\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}, \tilde{A}^{a^+}_{ij}$ and $\tilde{B}^{b^+}_{ij}$. We consider $\alpha_j = \tilde{A}^{a^-}_{ij}$ or $\tilde{A}^{a^+}_{ij}$ only, as the remaining cases are similar to this one. If $\alpha_j = \tilde{A}^{a^-}_{ij}$ then, $\tilde{B}^{b^-}_{ij} \leqslant \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^-}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$ and if $\beta_j = \tilde{B}^{b^+}_{ij}$, then $\tilde{A}^{a^-}_{ij} = \alpha_j = \max\{\lambda^a_{ij}, \mu^b_{ij}\} = \beta_j = \tilde{B}^{b^+}_{ij}$. Thus $\tilde{B}^{b^-}_{ij} \leqslant \tilde{B}^{b^+}_{ij} = \max\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{A}^{a^-}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$, which implies that

$$\max\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{B}^{b^+}_{ij} = (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+.$$

Hence

$$\max\{\lambda^a_{ij}, \mu^b_{ij}\} \notin \left((\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+\right) \text{ and}$$
$$(\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^- \leqslant \max\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+.$$

If $\alpha_j = \tilde{A}^{a^+}_{ij}$, then $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$ and so $\max\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{A}^{a^+}_{ij} = (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+$. Hence

$$\max\{\lambda^a_{ij}, \mu^b_{ij}\} \notin \left((\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+\right) \text{ and}$$
$$(\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^- \leqslant \max\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+.$$

Consequently, we note that $A^{\boxplus} \wedge_R B^{\boxplus}$ is both ECSM and ICSM. ■

THEOREM 4.18 *Let $A^{\boxplus}, B^{\boxplus} \in ECSM_{m\times n}$, such that* $\min\left\{\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\} \geqslant \min\{\lambda^a_{ij}, \mu^b_{ij}\} >$ $\max\left\{\min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}$ *for all i, j. Then, $A^{\boxplus} \wedge_P B^{\boxplus}$ is an ECSM.*

*Proof* Consider $A^{\boxplus} \wedge_P B^{\boxplus} = C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}^c_{ij}, \gamma^c_{ij} \right\rangle$, where $\tilde{C}^c_{ij} = \min\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}$ and $\gamma^c_{ij} = \min\{\lambda^a_{ij}, \mu^b_{ij}\}$. For each $e_j \in E$, take,

$$\alpha_j = \min\left\{\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}$$
$$\beta_j = \max\left\{\min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}.$$

Then $\alpha_j$ is one of $\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}, \tilde{A}^{a^+}_{ij}$ and $\tilde{B}^{b^+}_{ij}$. Consider $\alpha_j = \tilde{A}^{a^-}_{ij}$ or $\tilde{A}^{a^+}_{ij}$ only, for the remaining cases, it is similar to this case. If $\alpha_j = \tilde{A}^{a^-}_{ij}$, then $\tilde{B}^{b^-}_{ij} \leqslant \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^-}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$. If $\beta_j = \tilde{B}^{b^+}_{ij}$, then $\tilde{B}^{b^-}_{ij} = (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^- \leqslant (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+ = \tilde{B}^{b^+}_{ij} = \beta_j < \min\{\lambda^a_{ij}, \mu^b_{ij}\}$. Hence

$$\min\{\lambda^a_{ij}, \mu^b_{ij}\} \notin \left((\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+\right).$$

If $\alpha_j = \tilde{A}^{a^+}_{ij}$, then $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$ and so $\beta_j = \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\}$. Assume that $\beta_j = \tilde{A}^{a^-}_{ij}$, then $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \min\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$. From the given inequality, we have

$$\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \min\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij} \text{ or}$$
$$\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \min\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}.$$

For the case $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \min\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$, it is a contradiction to the fact that $A^{\boxplus}$ and $B^{\boxplus}$ are ECSM. For the case $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \min\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$, we have $\min\{\lambda^a_{ij}, \mu^b_{ij}\} \notin \left((\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+\right)$.
Since $\min\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{A}^{a^+}_{ij} = (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+$. Assume that $\beta_j = \tilde{B}^{b^-}_{ij}$.
Then $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \min\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$. From the given inequality,

$$\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \min\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij} \text{ or}$$
$$\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \min\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}.$$

For the case $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \min\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$, it contradicts to the fact that $A^{\boxplus}$ and $B^{\boxplus}$ are $ECSM_s$. For the case $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \min\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$, we get

$$\min\{\lambda^a_{ij}, \mu^b_{ij}\} \notin \left((\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+\right).$$

Since $\min\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{A}^{a^+}_{ij} = (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+$.
Hence, $A^{\boxplus} \wedge_P B^{\boxplus}$ is an ECSM. ■

The following example shows that for two $ECSM_s$, $A^{\boxplus}$ and $B^{\boxplus}$ which satisfy the condition $\min\left\{\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\} \geqslant \min(\lambda^a_{ij}, \mu^b_{ij}) = \max\left\{\min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}$, for all $i, j$. Then $A^{\boxplus} \wedge_P B^{\boxplus}$ may not be an ECSM.

*Example 4.19* Let $A^{\boxplus} = \begin{bmatrix} \langle [0.3, 0.7], 0.3 \rangle & \langle [0.2, 0.6], 0.7 \rangle \\ \langle [0.2, 0.6], 0.9 \rangle & \langle [0.1, 0.8], 0.1 \rangle \end{bmatrix}$ and
$B^{\boxplus} = \begin{bmatrix} \langle [0.2, 0.6], 0.7 \rangle & \langle [0.3, 0.7], 0.3 \rangle \\ \langle [0.4, 0.7], 0.4 \rangle & \langle [0.1, 0.5], 0.7 \rangle \end{bmatrix}$ satisfy the condition.
But

$$A^{\boxplus} \wedge_P B^{\boxplus} = \begin{bmatrix} \langle [0.2, 0.6], 0.3 \rangle & \langle [0.2, 0.6], 0.3 \rangle \\ \langle [0.2, 0.6], 0.4 \rangle & \langle [0.1, 0.5], 0.1 \rangle \end{bmatrix} \text{ is not an ECSM.}$$

THEOREM 4.20 *Let* $A^{\boxplus}, B^{\boxplus} \in ECSM_{m\times n}$, *such that*

$\min\left\{\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\} > \max\{\lambda^a_{ij}, \mu^b_{ij}\} \geqslant$
$\max\left\{\min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}$, *for all i, j. Then,* $A^{\boxplus} \vee_P B^{\boxplus}$ *is an ECSM.*

*Proof* Consider $A^{\boxplus} \vee_P B^{\boxplus} = C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}^c_{ij}, \gamma^c_{ij} \right\rangle$, where $\tilde{C}^c_{ij} = \max\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}$ and $\gamma^c_{ij} = \max\{\lambda^a_{ij}, \mu^b_{ij}\}$. For each $e_j \in E$, take,

$$\alpha_j = \min\left\{\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}$$
$$\beta_j = \max\left\{\min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}.$$

Then $\alpha_j$ is one of $\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}, \tilde{A}^{a^+}_{ij}$ and $\tilde{B}^{b^+}_{ij}$. We consider $\alpha_j = \tilde{A}^{a^-}_{ij}$ or $\tilde{A}^{a^+}_{ij}$ only. For the remaining cases, it is similar to this case. If $\alpha_j = \tilde{A}^{a^-}_{ij}$, then $\tilde{B}^{b^-}_{ij} \leqslant \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^-}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$ and so $\beta_j = \tilde{B}^{b^+}_{ij}$. Thus $(\tilde{A^a_{ij}} \cup \tilde{B^b_{ij}})^- = \tilde{A}^{a^-}_{ij} = \alpha_j > \max\{\lambda^a_{ij}, \mu^b_{ij}\}$ and hence

$$\max\{\lambda^a_{ij}, \mu^b_{ij}\} \notin \left((\tilde{A^a_{ij}} \cup \tilde{B^b_{ij}})^-, (\tilde{A^a_{ij}} \cup \tilde{B^b_{ij}})^+\right).$$

If $\alpha_j = \tilde{A}^{a^+}_{ij}$ then $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$ and so $\beta_j = \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\}$.
Assume that $\beta_j = \tilde{A}^{a^-}_{ij}$. Then $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} \leqslant \max\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$.
From the given inequality, we have

$$\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \max\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij} \text{ or}$$
$$\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} = \max\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}.$$

For the case $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \max(\lambda^a_{ij}, \mu^b_{ij}) < \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$, it is a contradicts to the fact that $A^{\boxplus}$ and $B^{\boxplus}$ are ECSM. For the case $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} = \max(\lambda^a_{ij}, \mu^b_{ij}) \leqslant \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$, we have

$$\max(\lambda^a_{ij}, \mu^b_{ij}) \notin \left((\tilde{A^a_{ij}} \cup \tilde{B^b_{ij}})^-_{ij}, (\tilde{A^a_{ij}} \cup \tilde{B^b_{ij}})^+\right).$$

Since $(\tilde{A^a_{ij}} \cup \tilde{B^b_{ij}})^- = \tilde{A}^{a^-}_{ij} = \max(\lambda^a_{ij}, \mu^b_{ij})$. Assume that $\beta_j = \tilde{B}^{b^-}_{ij}$.
Then $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} \leqslant \max(\lambda^a_{ij}, \mu^b_{ij}) \leqslant \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$. From the given inequality,

$$\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \max(\lambda^a_{ij}, \mu^b_{ij}) < \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij} \text{ or}$$
$$\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} = \max(\lambda^a_{ij}, \mu^b_{ij}) < \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}.$$

For the case $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \max(\lambda^a_{ij}, \mu^b_{ij}) < \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$, it contradicts to the fact that $A^{\boxplus}$ and $B^{\boxplus}$ are $ECSM_s$. For the case $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} = \max(\lambda^a_{ij}, \mu^b_{ij}) \leqslant \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$, we get

$$\max(\lambda^a_{ij}, \mu^b_{ij}) \notin \left((\tilde{A^a_{ij}} \cup \tilde{B^b_{ij}})^-, (\tilde{A^a_{ij}} \cup \tilde{B^b_{ij}})^+\right).$$

Since $(\tilde{A^a_{ij}} \cup \tilde{B^b_{ij}})^+ = \tilde{B}^{b^-}_{ij} = \max(\lambda^a_{ij}, \mu^b_{ij})$.
Hence the $A^{\boxplus} \vee_P B^{\boxplus}$ is an ECSM. ■

Theorem 4.21 *Let* $A^{\boxplus}, B^{\boxplus} \in CSM_{m\times n}$, *such that*
$\min\left\{\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\} > \min(\lambda^a_{ij}, \mu^b_{ij}) \geqslant$
$\max\left\{\min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}$, *for i, j. Then,* $A^{\boxplus} \vee_R B^{\boxplus}$ *is an ECSM.*

*Proof* Consider $A^{\boxplus} \vee_R B^{\boxplus} = C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}^c_{ij}, \gamma^c_{ij} \right\rangle$, where $\tilde{C}^c_{ij} = \max\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}$ and $\gamma^c_{ij} = \min\{\lambda^a_{ij}, \mu^b_{ij}\}$. For each $e_j \in E$ take,

$$\alpha_j = \min\left\{\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}$$

$$\beta_j = \max\left\{\min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\}.$$

Then $\alpha_j$ is one of $\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}, \tilde{A}_{ij}^{a^+}$ and $\tilde{B}_{ij}^{b^+}$. We consider $\alpha_j = \tilde{B}_{ij}^{b^-}$ or$\tilde{B}_{ij}^{b^+}$ only, for the remaining cases, it is similar to this case. If $\alpha_j = \tilde{B}_{ij}^{b^-}$, then $\tilde{A}_{ij}^{a^-} \leqslant \tilde{A}_{ij}^{a^+} \leqslant \tilde{B}_{ij}^{b^-} \leqslant \tilde{B}_{ij}^{b^+}$ and so $\beta_j = \tilde{A}_{ij}^{a^+}$. Thus by given inequality,

$$(\tilde{A_{ij}^{a}} \cup \tilde{B_{ij}^{b}})^- = \tilde{B}_{ij}^{b^-} = \alpha_j > \min\{\lambda_{ij}^a, \mu_{ij}^b\}$$

and hence $\min\{\lambda_{ij}^a, \mu_{ij}^b\} \notin ((\tilde{A_{ij}^{a}} \cup \tilde{B_{ij}^{b}})^-, (\tilde{A_{ij}^{a}} \cup \tilde{B_{ij}^{b}})^+)$.
If $\alpha_j = \tilde{B}_{ij}^{b^+}$, then $\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$ and so $\beta_j = \max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}$. Assume that $\beta_j = \tilde{A}_{ij}^{a^-}$, then $\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^-} \leqslant \min\{\lambda_{ij}^a, \mu_{ij}^b\} < \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$.
From the given inequality, we have

$$\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^-} < \min\{\lambda_{ij}^a, \mu_{ij}^b\} < \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+} \text{ or}$$
$$\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^-} = \min\{\lambda_{ij}^a, \mu_{ij}^b\} \leqslant \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}.$$

For the case $\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^-} < \min\{\lambda_{ij}^a, \mu_{ij}^b\} < \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$, it is a contradicts to the fact that $A^{\boxplus}$ and $B^{\boxplus}$ are ECSM. For the case $\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^-} = \min\{\lambda_{ij}^a, \mu_{ij}^b\} \leqslant \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$, we have

$$\min\{\lambda_{ij}^a, \mu_{ij}^b\} \notin \left((\tilde{A_{ij}^{a}} \cup \tilde{B_{ij}^{b}})^-, (\tilde{A_{ij}^{a}} \cup \tilde{B_{ij}^{b}})^+\right).$$

Since $(\tilde{A_{ij}} \cup \tilde{B_{ij}})^- = \tilde{A}_{ij}^{a^-} = \min\{\lambda_{ij}^a, \mu_{ij}^b\}$. Assume that $\beta_j = \tilde{B}_{ij}^{b^-}$. Then $\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^-} \leqslant \min\{\lambda_{ij}^a, \mu_{ij}^b\} \leqslant \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$. From the given inequality, we have $\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^-} < \min\{\lambda_{ij}^a, \mu_{ij}^b\} < \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$ or $\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^-} = \min\{\lambda_{ij}^a, \mu_{ij}^b\} < \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$. For the case $\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^-} < \min\{\lambda_{ij}^a, \mu_{ij}^b\} < \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$, it contradicts to the fact that $A^{\boxplus}$ and $B^{\boxplus}$ are ECSMs. For the case $\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^-} = \min\{\lambda_{ij}^a, \mu_{ij}^b\} \leqslant \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$, we get

$$\min\{\lambda_{ij}^a, \mu_{ij}^b\} \notin \left((\tilde{A_{ij}^{a}} \cup \tilde{B_{ij}^{b}})^-, (\tilde{A_{ij}^{a}} \cup \tilde{B_{ij}^{b}})^+\right).$$

Since$(\tilde{A}_{ij}^{a} \cap \tilde{B}_{ij}^{b})^- = \tilde{B}_{ij}^{b^-} = \min\{\lambda_{ij}^a, \mu_{ij}^b\}$.
Hence the $A^{\boxplus} \vee_R B^{\boxplus}$ is an ECSM. ■

The following example shows that for two $ECSM_s$, $A^{\boxplus}$ and $B^{\boxplus}$ which satisfy the condition $\min\left\{\max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\} = \min\{\lambda_{ij}^a, \mu_{ij}^b\} > \max\left\{\min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\}$, for all i, j. Then $A^{\boxplus} \vee_R B^{\boxplus}$ may not be an ECSM.

*Example 4.22* Let $A^{\boxplus} = \begin{bmatrix} \langle [0.3, 0.6], 0.6\rangle & \langle [0.1, 0.8], 0.9\rangle \\ \langle [0.4, 0.5], 0.5\rangle & \langle [0.3, 0.6], 0.6\rangle \end{bmatrix}$ and
$B^{\boxplus} = \begin{bmatrix} \langle [0.1, 0.7], 0.8\rangle & \langle [0.2, 0.7], 0.7\rangle \\ \langle [0.3, 0.8], 0.9\rangle & \langle [0.2, 0.7], 0.8\rangle \end{bmatrix}$ satisfy the above condition.
But

$$A^{\boxplus} \vee_R B^{\boxplus} = \begin{bmatrix} \langle [0.3, 0.7], 0.6\rangle & \langle [0.2, 0.8], 0.7\rangle \\ \langle [0.4, 0.8], 0.5\rangle & \langle [0.3, 0.7], 0.6\rangle \end{bmatrix} \text{ is not an ECSM.}$$

Theorem 4.23 *Let $A^{\boxplus}, B^{\boxplus} \in CSM_{m\times n}$, such that*
$\min\left\{\max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\} \geqslant \max\{\lambda_{ij}^a, \mu_{ij}^b\} >$
$\max\left\{\min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\}$ *for all i, j. Then, $A^{\boxplus} \wedge_R B^{\boxplus}$ is an ECSM.*

*Proof* Consider $A^{\boxplus} \wedge_R B^{\boxplus} = C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}^c_{ij}, \gamma^c_{ij} \right\rangle$, where $\tilde{C}^c_{ij} = \min\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}$ and $\gamma^c_{ij} = \max\{\lambda^a_{ij}, \mu^b_{ij}\}$. For each $e_j \in E$, take

$$\alpha_j = \min\left\{\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}$$
$$\beta_j = \max\left\{\min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}.$$

Then $\alpha_j$ is one of $\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}, \tilde{A}^{a^+}_{ij}$ and $\tilde{B}^{b^+}_{ij}$. We consider $\alpha_j = \tilde{A}^{a^-}_{ij}$ or$\tilde{A}^{a^+}_{ij}$ only, for the remaining cases, it is similar to this case. If $\alpha_j = \tilde{B}^{b^-}_{ij}$, then $\tilde{A}^{a^-}_{ij} \leqslant \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^-}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$ and so $\beta_j = \tilde{A}^{a^+}_{ij}$. Thus $\tilde{A}^{a^-}_{ij} = (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^- \leqslant (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+ = \beta_j < \max\{\lambda^a_{ij}, \mu^b_{ij}\}$ and hence

$$\max\{\lambda^a_{ij}, \mu^b_{ij}\} \notin \left((\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+\right).$$

If $\alpha_j = \tilde{B}^{b^+}_{ij}$ then $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$ and so $\beta_j = \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}$.
Assume that $\beta_j = \tilde{A}^{a^-}_{ij}$. Then $\tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^a_{ij} < \max\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$.
From the given inequality, we have

$$\tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^a_{ij} < \max\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij} \text{ or}$$
$$\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \max\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}.$$

For the case $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \max\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$, it is a contradiction to the fact that $A^{\boxplus}$ and $B^{\boxplus}$ are ECSM. For the case $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \max\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$, we have

$$\max\{\lambda^a_{ij}, \mu^b_{ij}\} \notin \left((\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+\right).$$

Since $\max\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{A}^{a^+}_{ij} = (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+$. Assume that $\beta_j = \tilde{B}^{b^-}_{ij}$. Then $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \max\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$. From the given inequality, we have

$$\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \max\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij} \text{ or}$$
$$\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \max\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}.$$

For the case $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \max\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$, it contradicts to the fact that $A^{\boxplus}$ and $B^{\boxplus}$ are ECSMs. For the case $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \max\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$, we get $\max\{\lambda^a_{ij}, \mu^b_{ij}\} \notin ((\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+)$.
Since $\max\{\lambda^a_{ij}, \mu^b_{ij}\} = \tilde{A}^{a^+}_{ij} = (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+$.
Hence the $A^{\boxplus} \wedge_R B^{\boxplus}$ is an ECSM. ■

The following example shows that for two $ECSM_s$, $A^{\boxplus}$ and $B^{\boxplus}$ which satisfy the condition $\min\left\{\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\} > \max\{\lambda^a_{ij}, \mu^b_{ij}\} = \max\left\{\min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}$, for all i, j. Then $A^{\boxplus} \wedge_R B^{\boxplus}$ may not be an ECSM.

*Example 4.24* Let $A^{\boxplus}= \begin{bmatrix} \langle[0.7, 0.9], 0.7\rangle & \langle[0.4, 0.7], 0.2\rangle \\ \langle[0.3, 0.6], 0.3\rangle & \langle[0.6, 0.7], 0.5\rangle \end{bmatrix}$ and

$B^{\boxplus}= \begin{bmatrix} \langle[0.6, 0.8], 0.4\rangle & \langle[0.5, 0.8], 0.5\rangle \\ \langle[0.2, 0.4], 0.1\rangle & \langle[0.8, 1], 0.7\rangle \end{bmatrix}$ satisfy the above condition.
But

$$A^{\boxplus} \wedge_R B^{\boxplus} = \begin{bmatrix} \langle[0.6, 0.8], 0.7\rangle & \langle[0.4, 0.7], 0.5\rangle \\ \langle[0.2, 0.4], 0.3\rangle & \langle[0.6, 0.7], 0.7\rangle \end{bmatrix} \text{ is not an ECSM.}$$

THEOREM 4.25 *Let $A^{\boxplus}, B^{\boxplus} \in ECSM_{m\times n}$, such that*

$\min(\lambda^a_{ij},\mu^b_{ij}) \in \left[\min\left\{\max\{\tilde{A}^{a^+}_{ij},\tilde{B}^{b^-}_{ij}\},\max\{\tilde{A}^{a^-}_{ij},\tilde{B}^{b^+}_{ij}\}\right\},\max\left\{\min\{\tilde{A}^{a^+}_{ij},\tilde{B}^{b^-}_{ij}\},\min\{\tilde{A}^{a^-}_{ij},\tilde{B}^{b^+}_{ij}\}\right\}\right)$
*for all i, j. Then* $A^{\boxplus} \wedge_P B^{\boxplus} \in ECSM_{m\times n}$.

*Proof* Consider $A^{\boxplus} \wedge_P B^{\boxplus} = C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}^c_{ij}, \gamma^c_{ij}\right\rangle$, where $\tilde{C}^c_{ij} = \min\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}$ and $\gamma^c_{ij} = \min\{\lambda^a_{ij},\mu^b_{ij}\}$. For each $e_j \in E$, take

$$\alpha_j = \min\left\{\max\{\tilde{A}^{a^+}_{ij},\tilde{B}^{b^-}_{ij}\},\max\{\tilde{A}^{a^-}_{ij},\tilde{B}^{b^+}_{ij}\}\right\}$$
$$\beta_j = \max\left\{\min\{\tilde{A}^{a^+}_{ij},\tilde{B}^{b^-}_{ij}\},\min\{\tilde{A}^{a^-}_{ij},\tilde{B}^{b^+}_{ij}\}\right\}.$$

Then $\alpha_j$ is one $\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}, \tilde{A}^{a^+}_{ij}$ and $\tilde{B}^{b^+}_{ij}$. Now consider,$\alpha_j = \tilde{A}^{a^-}_{ij}$ or $\tilde{A}^{a^+}_{ij}$ only, as the remaining cases are similar to this one. If $\alpha_j = \tilde{A}^{a^-}_{ij}$, then $\tilde{B}^{b^-}_{ij} \leqslant \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^-}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$ and so $\beta_j = \tilde{B}^{b^+}_{ij}$, thus $\tilde{B}^{b^-}_{ij} = (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^- \leqslant (\tilde{A}^a_{ij} \cap \tilde{A}^a_{ij})^+ = \tilde{B}^{b^+}_{ij} = \beta_j < \min\{\lambda^a_{ij},\mu^b_{ij}\}$. Hence

$$\min\{\lambda^a_{ij},\mu^b_{ij}\} \notin \left((\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+\right).$$

If $\alpha_j = \tilde{A}^{a^+}_{ij}$, then $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$ and so $\beta_j = \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\}$. Assume that $\beta_j = \tilde{A}^{a^-}_{ij}$, then $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \min\{\lambda^a_{ij},\mu^b_{ij}\} \leqslant \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$. So from this,

$$\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \min\{\lambda^a_{ij},\mu^b_{ij}\} < \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij} \text{ or}$$
$$\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \min\{\lambda^a_{ij},\mu^b_{ij}\} = \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}.$$

For the case, $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \min\{\lambda^a_{ij},\mu^b_{ij}\} < \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$, it is contradiction to the fact that, $A^{\boxplus}$ and $B^{\boxplus}$ are $ECSM_{m\times n}$. For the case $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \min\{\lambda^a_{ij},\mu^b_{ij}\} = \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$, we have $\min\{\lambda^a_{ij},\mu^b_{ij}\} \notin \left((\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+\right)$ because $\min\{\lambda^a_{ij},\mu^b_{ij}\} = \tilde{A}^{a^+}_{ij} = (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+$.
Again assume that $\beta_j = \tilde{B}^{b^-}_{ij}$, then $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \min\{\lambda^a_{ij},\mu^b_{ij}\} \leqslant \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$. From this,

$$\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \min\{\lambda^a_{ij},\mu^b_{ij}\} < \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}. \text{ or}$$
$$\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \min\{\lambda^a_{ij},\mu^b_{ij}\} = \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}.$$

For the case $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \min\{\lambda^a_{ij},\mu^b_{ij}\} < \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$ it is contradiction to the fact that $A^{\boxplus}$ and $B^{\boxplus}$ are $ECSM_{m\times n}$. And if we take the case, $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \min\{\lambda^a_{ij},\mu^b_{ij}\} = \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$ we get, $\min\{\lambda^a_{ij},\mu^b_{ij}\} \notin \left((\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+\right)$

because, $\min\{\lambda^a_{ij},\mu^b_{ij}\} = \tilde{A}^{a+}_{ij} = (\tilde{A}^a_{ij} \cap \tilde{B}^b_{ij})^+$.
Hence in all the cases,$A^{\boxplus} \wedge_P B^{\boxplus} \in ECSM_{m\times n}$. ■

The following example shows that for two ECSMs, $A^{\boxplus}$ and $B^{\boxplus}$ which satisfy the condition.
$\min(\lambda^a_{ij},\mu^b_{ij}) \notin \left[\min\left\{\max\{\tilde{A}^{a^+}_{ij},\tilde{B}^{b^-}_{ij}\},\max\{\tilde{A}^{a^-}_{ij},\tilde{B}^{b^+}_{ij}\}\right\},\max\left\{\min\{\tilde{A}^{a^+}_{ij},\tilde{B}^{b^-}_{ij}\},\min\{\tilde{A}^{a^-}_{ij},\tilde{B}^{b^+}_{ij}\}\right\}\right)$
for all i, j. Then $A^{\boxplus} \wedge_P B^{\boxplus}$ may not be an ECSM.

*Example 4.26* Let $A^{\boxplus} = \begin{bmatrix} \langle[0.2,0.3],0.2\rangle & \langle[0.4,0.7],0.2\rangle \\ \langle[0.5,0.9],0.5\rangle & \langle[0.1,0.5],0.9\rangle \end{bmatrix}$
and $B^{\boxplus} = \begin{bmatrix} \langle[0.1,0.3],0.7\rangle & \langle[0.4,0.6],0.3\rangle \\ \langle[0.4,0.9],1\rangle & \langle[0.8,1],0.5\rangle \end{bmatrix}$
Then

$$A^{\boxplus} \wedge_P B^{\boxplus} = \begin{bmatrix} \langle[0.1,0.3],0.2\rangle & \langle[0.4,0.6],0.2\rangle \\ \langle[0.4,0.9],0.5\rangle & \langle[0.1,0.5],0.5\rangle \end{bmatrix} \text{ is not an ECSM.}$$

Theorem 4.27 *Let* $A^{\boxplus}, B^{\boxplus} \in ECSM_{m\times n}$, *such that*

$$\max\{\lambda_{ij}^a, \mu_{ij}^b\} \in \left( \min\left\{\max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\}, \max\left\{\min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\} \right]$$

*for all i, j. Then* $A^{\boxplus} \vee_P B^{\boxplus} \in ECSM_{m\times n}$.

*Proof* Consider $A^{\boxplus} \vee_P B^{\boxplus} = C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle$, where $\tilde{C}_{ij}^c = \max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}$ and $\gamma_{ij}^c = \max\{\lambda_{ij}^a, \mu_{ij}^b\}$. For each $e_j \in E$, take

$$\alpha_j = \min\left\{\max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\}$$
$$\beta_j = \max\left\{\min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\}.$$

Then $\alpha_j$ is one $\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}, \tilde{A}_{ij}^{a^+}$ and $\tilde{B}_{ij}^{b^+}$. Now consider,$\alpha_j = \tilde{A}_{ij}^{a^-}$ or $\tilde{A}_{ij}^{a^+}$ only, as the remaining cases are similar to this one. If $\alpha_j = \tilde{A}_{ij}^{a^-}$, then $\tilde{B}_{ij}^{b^-} \leqslant \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^-} \leqslant \tilde{A}_{ij}^{a^+}$ and so $\beta_j = \tilde{B}_{ij}^{b^+}$. Thus $(\tilde{A}_{ij}^a \cup \tilde{B}_{ij}^b)^- = \tilde{A}_{ij}^{a^-} = \alpha_j > \max\{\lambda_{ij}^a, \mu_{ij}^b\}$. Hence

$$\max\{\lambda_{ij}^a, \mu_{ij}^b\} \notin \left((\tilde{A}_{ij}^a \cup \tilde{B}_{ij}^b)^-, (\tilde{A}_{ij}^a \cup \tilde{B}_{ij}^b)^+\right).$$

If $\alpha_j = \tilde{A}_{ij}^{a^+}$, then $\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^+} \leqslant \tilde{B}_{ij}^{b^+}$ and so $\beta_j = \max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}$. Assume that $\beta_j = \tilde{A}_{ij}^{a^-}$, then $\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^-} \leqslant \max\{\lambda_{ij}^a, \mu_{ij}^b\} < \tilde{A}_{ij}^{a^+} \leqslant \tilde{B}_{ij}^{b^+}$. So from this we get,

$$\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^-} < \max\{\lambda_{ij}^a, \mu_{ij}^b\} < \tilde{A}_{ij}^{a^+} \leqslant \tilde{B}_{ij}^{b^+} \text{ or}$$
$$\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^-} = \max\{\lambda_{ij}^a, \mu_{ij}^b\} \leqslant \tilde{A}_{ij}^{a^+} \leqslant \tilde{B}_{ij}^{b^+}.$$

For the cases, $\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^-} < \max\{\lambda_{ij}^a, \mu_{ij}^b\} < \tilde{A}_{ij}^{a^+} \leqslant \tilde{B}_{ij}^{b^+}$, it is contradiction to the fact that, $A^{\boxplus}$ and $B^{\boxplus}$ are $ECSM_{m\times n}$. For the case $\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^-} < \max\{\lambda_{ij}^a, \mu_{ij}^b\} \leqslant \tilde{A}_{ij}^{a^+} \leqslant \tilde{B}_{ij}^{b^+}$, we have $\max\{\lambda_{ij}^a, \mu_{ij}^b\} \notin \left((\tilde{A}_{ij}^a \cap \tilde{B}_{ij}^b)^-, (\tilde{A}_{ij}^a \cap \tilde{A}_{ij}^a)^+\right)$ because $(\tilde{A}_{ij}^a \cup \tilde{A}_{ij}^a)^- = \tilde{A}_{ij}^{a^-} = \max\{\lambda_{ij}^a, \mu_{ij}^b\}$.

Again assume that $\beta_j = \tilde{B}_{ij}^{b^-}$, then $\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^-} < \max\{\lambda_{ij}^a, \mu_{ij}^b\} \leqslant \tilde{A}_{ij}^{a^+} \leqslant \tilde{B}_{ij}^{b^+}$, so from this we get,

$$\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^-} < \max\{\lambda_{ij}^a, \mu_{ij}^b\} < \tilde{A}_{ij}^{a^+} \leqslant \tilde{B}_{ij}^{b^+} \text{ or}$$
$$\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^-} < \max\{\lambda_{ij}^a, \mu_{ij}^b\} = \tilde{A}_{ij}^{a^+} \leqslant \tilde{B}_{ij}^{b^+}.$$

For the case $\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^-} < \max\{\lambda_{ij}^a, \mu_{ij}^b\} < \tilde{A}_{ij}^{a^+} \leqslant \tilde{B}_{ij}^{b^+}$, it is contradiction to the fact that $A^{\boxplus}$ and $B^{\boxplus}$ are $ECSM_{m\times n}$. And if we take the case, $\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^-} < \max\{\lambda_{ij}^a, \mu_{ij}^b\} \leqslant \tilde{A}_{ij}^{a^+} \leqslant \tilde{B}_{ij}^{b^+}$ we get, $\max\{\lambda_{ij}^a, \mu_{ij}^b\} \notin \left((\tilde{A}_{ij}^a \cup \tilde{B}_{ij}^b)^-, (\tilde{A}_{ij}^a \cup \tilde{B}_{ij}^b)^+\right)$ because, $(\tilde{A}_{ij}^a \cup \tilde{B}_{ij}^b)^- = \tilde{B}_{ij}^{b^-} = \max\{\lambda_{ij}^a, \mu_{ij}^b\}$.

Hence in all the cases,$A^{\boxplus} \vee_P B^{\boxplus} \in ECSM_{m\times n}$. ■

The following example shows that for two ECSMs, $A^{\boxplus}$ and $B^{\boxplus}$ which satisfy the condition.

$$\min\{\lambda_{ij}^a, \mu_{ij}^b\} \notin \left( \min\left\{\max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\}, \max\left\{\min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\} \right]$$

for all i, j. Then $A^{\boxplus} \vee_P B^{\boxplus}$ may not be an ECSM.

*Example 4.28* Let $A^{\boxplus} = \begin{bmatrix} \langle[0.2, 0.3], 0.2\rangle & \langle[0.4, 0.7], 0.2\rangle \\ \langle[0.5, 0.9], 0.5\rangle & \langle[0.1, 0.5], 0.9\rangle \end{bmatrix}$ and

$B^{\boxplus} = \begin{bmatrix} \langle[0.1, 0.3], 0.7\rangle & \langle[0.4, 0.6], 0.3\rangle \\ \langle[0.4, 0.9], 1\rangle & \langle[0.8, 1], 0.5\rangle \end{bmatrix}$. Then

$$A^{\boxplus} \vee_P B^{\boxplus} = \begin{bmatrix} \langle[0.2, 0.3], 0.7\rangle & \langle[0.4, 0.7], 0.3\rangle \\ \langle[0.5, 0.9], 1\rangle & \langle[0.8, 1], 0.9\rangle \end{bmatrix} \text{ is not an ECSM.}$$

THEOREM 4.29 *Let* $A^{\boxplus}, B^{\boxplus} \in ECSM_{m\times n}$, *such that* $\min\{\lambda^a_{ij}, \mu^b_{ij}\} \in \left(\min\left\{\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}, \max\left\{\min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}\right]$. *Then* $A^{\boxplus} \vee_R B^{\boxplus} \in ECSM_{m\times n}$.

*Proof* Consider $A^{\boxplus} \vee_R B^{\boxplus} = C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}^c_{ij}, \gamma^c_{ij} \right\rangle$, where $\tilde{C}^c_{ij} = \max\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}$ and $\gamma^c_{ij} = \min\{\lambda^a_{ij}, \mu^b_{ij}\}$. For each $e_j \in E$, take

$$\alpha_j = \min\left\{\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}$$
$$\beta_j = \max\left\{\min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}.$$

Then $\alpha_j$ is one $\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}, \tilde{A}^{a^+}_{ij}$ and $\tilde{B}^{b^+}_{ij}$. Now consider,$\alpha_j = \tilde{B}^{b^-}_{ij}$ or $\tilde{B}^{b^+}_{ij}$ only, as the remaining cases are similar to this one. If $\alpha_j = \tilde{B}^{b^-}_{ij}$, then $\tilde{A}^{a^-}_{ij} \leqslant \tilde{A}^{a^+}_{ij} \leqslant \tilde{B}^{b^-}_{ij} \leqslant \tilde{B}^{b^+}_{ij}$ and so $\beta_j = \tilde{A}^{a^+}_{ij}$. Thus $(\tilde{A}^a_{ij} \cup \tilde{B}^b_{ij})^- = \tilde{B}^{b^-}_{ij} = \alpha_j > \min\{\lambda^a_{ij}, \mu^b_{ij}\}$. Hence

$$\min\{\lambda^a_{ij}, \mu^b_{ij}\} \notin \left((\tilde{A}^a_{ij} \cup \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cup \tilde{B}^b_{ij})^+\right).$$

If $\alpha_j = \tilde{B}^{b^+}_{ij}$, then $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$ and so $\beta_j = \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\}$. Assume that $\beta_j = \tilde{A}^{a^-}_{ij}$, then $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \min\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$. Then,

$$\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \min\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij} \text{ or}$$
$$\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} = \min\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}.$$

For the cases, $\tilde{B}^{b^-}_{ij} \leqslant \tilde{A}^{a^-}_{ij} < \min\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$, it is contradiction to the fact that, $A^{\boxplus}$ and $B^{\boxplus}$ are $ECBM_{m\times n}$. For the case $\tilde{B}^{b^-}_{ij} < \tilde{A}^{a^-}_{ij} = \min\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$, we have $\min\{\lambda^a_{ij}, \mu^b_{ij}\} \notin \left((\tilde{A}^a_{ij} \cup \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cup \tilde{B}^b_{ij})^+\right)$ because $(\tilde{A}^a_{ij} \cup \tilde{B}^b_{ij})^- = \tilde{A}^{a^-}_{ij} = \min\{\lambda^a_{ij}, \mu^b_{ij}\}$.

Again assume that $\beta_j = \tilde{B}^{b^-}_{ij}$, then $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} \leqslant \min\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$. From this we get,

$$\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \min\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij} \text{ or}$$
$$\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} = \min\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}.$$

For the case $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} < \min\{\lambda^a_{ij}, \mu^b_{ij}\} < \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$ it is contradiction to the fact that $A^{\boxplus}$ and $B^{\boxplus}$ are $ECSM_{m\times n}$. And if the case, $\tilde{A}^{a^-}_{ij} \leqslant \tilde{B}^{b^-}_{ij} = \min\{\lambda^a_{ij}, \mu^b_{ij}\} \leqslant \tilde{B}^{b^+}_{ij} \leqslant \tilde{A}^{a^+}_{ij}$ is taken, then $\min\{\lambda^a_{ij}, \mu^b_{ij}\} \notin \left((\tilde{A}^a_{ij} \cup \tilde{B}^b_{ij})^-, (\tilde{A}^a_{ij} \cup \tilde{B}^b_{ij})^+\right)$ because, $(\tilde{A}^a_{ij} \cup \tilde{B}^b_{ij})^- = \tilde{B}^{b^-}_{ij} = \min\{\lambda^a_{ij}, \mu^b_{ij}\}$.

Hence in all the cases, $A^{\boxplus} \vee_R B^{\boxplus}$ is an $ECSM_{m\times n}$. ■

The following example shows that for two $ECSM_s$, $A^{\boxplus}$ and $B^{\boxplus}$ which satisfy the condition. $\min\{\lambda^a_{ij}, \mu^b_{ij}\} \notin \left[\min\left\{\max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}, \max\left\{\min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^+}_{ij}\}\right\}\right)$ for all i, j. Then $A^{\boxplus} \vee_R B^{\boxplus}$ may not be an ECSM.

*Example 4.30* Let $A^{\boxplus} = \begin{bmatrix} \langle[0.3, 0.7], 0.9\rangle & \langle[0.4, 0.8], 0.9\rangle \\ \langle[0.1, 0.3], 0.4\rangle & \langle[0.2, 0.4], 0.7\rangle \end{bmatrix}$ and

$B^{\boxplus} = \begin{bmatrix} \langle[0.2, 0.4], 0.6\rangle & \langle[0.6, 0.7], 0.5\rangle \\ \langle[0.1, 0.9], 1\rangle & \langle[0.5, 0.9], 0.2\rangle \end{bmatrix}$. Then

$$A^{\boxplus} \vee_R B^{\boxplus} = \begin{bmatrix} \langle[0.3, 0.7], 0.6\rangle & \langle[0.6, 0.8], 0.5\rangle \\ \langle[0.1, 0.9], 0.2\rangle & \langle[0.5, 0.9], 0.2\rangle \end{bmatrix} \text{ is not an ECSM.}$$

Theorem 4.31 *Let $A^{\boxplus}, B^{\boxplus} \in ECSM_{m\times n}$, such that*
$\max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} \in \left[\min\left\{\max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\}, \max\left\{\min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\}\right)$
*for all i, j. Then $A^{\boxplus} \wedge_R B^{\boxplus} \in ECSM_{m\times n}$.*

*Proof* Consider $A^{\boxplus} \wedge_R B^{\boxplus} = C^{\boxplus} = [c_{ij}] = \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle$, where $\tilde{C}_{ij}^{c} = \min\{\tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b}\}$ and $\gamma_{ij}^{c} = \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\}$. For each $e_j \in E$, take

$$\alpha_j = \min\left\{\max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\}$$
$$\beta_j = \max\left\{\min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\}.$$

Then $\alpha_j$ is one $\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}, \tilde{A}_{ij}^{a^+}$ and $\tilde{B}_{ij}^{b^+}$. Now consider,$\alpha_j = \tilde{B}_{ij}^{b^-}$ or $\tilde{B}_{ij}^{b^+}$ only, as the remaining cases are similar to this one. If $\alpha_j = \tilde{B}_{ij}^{b^-}$, then $\tilde{A}_{ij}^{a^-} \leqslant \tilde{A}_{ij}^{a^+} \leqslant \tilde{B}_{ij}^{b^-} \leqslant \tilde{B}_{ij}^{b^+}$ and so $\beta_j = \tilde{A}_{ij}^{a^+}$. Thus $(\tilde{A}_{ij}^{a} \cap \tilde{B}_{ij}^{b})^+ = \tilde{A}_{ij}^{a^+} = \beta_j < \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\}$. Hence

$$\max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} \notin \left((\tilde{A}_{ij}^{a} \cap \tilde{B}_{ij}^{b})^-, (\tilde{A}_{ij}^{a} \cap \tilde{B}_{ij}^{b})^+\right).$$

If $\alpha_j = \tilde{B}_{ij}^{b^+}$, then $\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$ and so $\beta_j = \max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}$. Assume that $\beta_j = \tilde{A}_{ij}^{a^-}$, then $\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^-} < \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} \leqslant \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$. So from this we get,

$$\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^-} < \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} < \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+} \text{ or}$$
$$\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^-} < \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} = \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}.$$

For the case, $\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^-} < \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} < \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$, it is contradiction to the fact that, $A^{\boxplus}$ and $B^{\boxplus}$ are $ECSM_{m\times n}$. For the case $\tilde{B}_{ij}^{b^-} \leqslant \tilde{A}_{ij}^{a^-} < \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} \leqslant \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$. Then $\max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} \notin \left((\tilde{A}_{ij}^{a} \cap \tilde{B}_{ij}^{b})^-, (\tilde{A}_{ij}^{a} \cap \tilde{B}_{ij}^{b})^+\right)$ because

$$(\tilde{A}_{ij}^{a} \cap \tilde{B}_{ij}^{b})^+ = \tilde{B}_{ij}^{b^+} = \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\}.$$

Again assume that $\beta_j = \tilde{B}_{ij}^{b^-}$, then $\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^-} < \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} \leqslant \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$. From this we get,

$$\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^-} < \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} < \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+} \text{ or}$$
$$\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^-} < \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} < \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}.$$

For the case $\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^-} < \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} < \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$ it is contradiction to the fact that $A^{\boxplus}$ and $B^{\boxplus}$ are $ECSM_{m\times n}$. And if the case, $\tilde{A}_{ij}^{a^-} \leqslant \tilde{B}_{ij}^{b^-} < \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} = \tilde{B}_{ij}^{b^+} \leqslant \tilde{A}_{ij}^{a^+}$, then $\max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} \notin \left((\tilde{A}_{ij}^{a} \cap \tilde{B}_{ij}^{b})^-, (\tilde{A}_{ij}^{a} \cap \tilde{B}_{ij}^{b})^+\right)$ because,

$$(\tilde{A}_{ij}^{a} \cap \tilde{B}_{ij}^{b})^+ = \tilde{B}_{ij}^{b^+} = \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\}.$$

Thus, $A^{\boxplus} \wedge_R B^{\boxplus}$ is an $ECSM_{m\times n}$. ■

The following example shows that for two $ECSM_s$, $A^{\boxplus}$ and $B^{\boxplus}$ which satisfy the condition.
$\max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} \notin \left(\min\left\{\max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\}, \max\left\{\min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^+}\}\right\}\right]$
for all i, j. Then $A^{\boxplus} \wedge_R B^{\boxplus}$ may not be an ECSM.

*Example 4.32* Let $A^{\boxplus} = \begin{bmatrix} \langle[0.3, 0.8], 0.9\rangle & \langle[0.4, 0.6], 0.8\rangle \\ \langle[0.4, 0.7], 1\rangle & \langle[0.2, 0.5], 0.6\rangle \end{bmatrix}$
and $B^{\boxplus} = \begin{bmatrix} \langle[0.4, 0.6], 0.7\rangle & \langle[0.1, 0.8], 0.4\rangle \\ \langle[0.2, 0.4], 0.6\rangle & \langle[0.4, 0.6], 0.9\rangle \end{bmatrix}$. Then

$$A^{\boxplus} \wedge_R B^{\boxplus} = \begin{bmatrix} \langle[0.4, 0.8], 0.7\rangle & \langle[0.4, 0.8], 0.4\rangle \\ \langle[0.4, 0.7], 0.6\rangle & \langle[0.4, 0.6], 0.6\rangle \end{bmatrix} \text{ is not an ECSM.}$$

## References

[1] Chen D., Tsang E.C.C., Yeung D.S., and Wang X.," The parameterization reduction of soft sets and its applications", Computers and Mathematices with Applications 57, (2009), 1547-1553.
[2] Jun Y.B., Kim C.S. and Yang K.O., "Cubic sets", Annals of fuzzy Mathematics and Informatics, 4, (2012), 83-98.
[3] Kim J.B., "Idempotent and Inverses in Fuzzy Matrices", Malaysian Math, 6(2), (1983), 57-61.
[4] Maji P.K., Biswas R. and Roy A.R., "Fuzzy soft sets", Journal of Fuzzy Mathematics, 9, (2001), 589-602.
[5] Maji P.K and Roy A.R., "Soft set theory", Computers and Mathematics with Applications 45, (2003), 555-562.
[6] Meenakshi A.R., " Fuzzy Matrix Theory and Applications", MJP Publishers, (2008).
[7] Molodstov D.A., Soft set theory-first results, Computers and Mathematics with Applications 37, (1999), 19-31.
[8] Muhiuddin G., Al-roqi A.M., "Cubic soft sets with applications in BCK/BCL-algebras", Annals of Fuzzy Mathematics and Informatics, In press.
[9] Pei D. and Miao D., "From soft sets to information systems", in Proceedings of the IEEE International Conference on Granular Computing, 2, (2005), 617-621.
[10] Thomason M.G., " Convergence power of fuzzy matrix", J. of Math. Anal. Appl. 57, (1977), 476-486.
[11] Zadeh L.A., "Fuzzy sets", Information and Control 8, (1965), 338-353.