The Length-Biased weighted Exponentiated Lomax Distribution

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Abstract:

In this paper, we introduce a new family of distributions called Length-Biased weighted Exponentiated Lomax Distribution (LBWELD). Some properties of this family will be discussed. The estimation of unknown parameters for LBWELD will be handled using Maximum Likelihood method. Finally, an application to real data sets is illustrated.

Key words: Weighted distribution - Exponentiated Lomax distribution – Maximum Likelihood Estimation – Information entropies – Moments.

1. Introduction

The first appear of the concept "weighted distributions" can be traced to Fisher (1934). Rao (1965), identified various situations that can be modeled by weighted distributions.

Let X be a non-negative random variable with probability density function (*pdf*) $g(x)$. The *pdf* of the weighted random variable X is given by

$$f(x)=\frac{w(x)g(x)}{E\left[w(X)\right]},\ x>0$$

where $w(x)$ be a non-negative weight function.

When $w(x)=x$, the distribution is called length-biased, whose *pdf* is

$$f(x)=\frac{xg(x)}{E(X)},\ x>0 \qquad (1)$$

The formula (1) is used by many authors. Shaban and Boudrissa (2007) discussed the Weibull length biased distribution with properties and estimation. The length biased weighted generalized Rayleigh distribution is introduced by Das and Roy (2011). Seenoi et al (2014) discussed the length biased exponentiated inverted Weibull distribution. The length biased weighted Lomax distribution, statistical properties and application is introduced by Afaq et al (2016).

A random variable X is said to have an exponentiated Lomax distribution with three parameters θ, λ and α if it's *pdf* is in the form (Abdul-Moniem and Abdel-Hameed (2012))

$$g(x)=\alpha\theta\lambda\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha-1}(1+\lambda x)^{-(\theta+1)};\ x>0,\ (\alpha,\theta \text{ and } \lambda>0). \quad (2)$$

The $E(X)$ corresponding (2) is

$$E(X)=\frac{\alpha}{\lambda}\left[B\left(1-\frac{1}{\theta},\alpha\right)-\frac{1}{\alpha}\right]=\frac{\alpha}{\lambda}D_1. \quad (3)$$

Where $D_j=B\left(1-\frac{j}{\theta},\alpha\right)-B\left(1-\frac{j-1}{\theta},\alpha\right),\quad j=1,2,3.$

2. The Length-Biased weighted Exponentiated Lomax Distribu-tion

Using (1), (2) and (3), we can define the *pdf* of length-biased weighted exponentiated Lomax distribution (LBWELD) as follows

$$f(x)=\frac{\alpha\theta\lambda x\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha-1}(1+\lambda x)^{-(\theta+1)}}{\frac{\alpha}{\lambda}D_1};\ x>0,\ (\alpha,\theta \text{ and } \lambda>0)$$

$$=\frac{\theta\lambda^2 x\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha-1}(1+\lambda x)^{-(\theta+1)}}{D_1};\ x>0,\ (\alpha,\theta \text{ and } \lambda>0). \quad (4)$$

We can get the *pdf* for length-biased weighted exponentiated Pareto (LBWEP), length-biased weighted Pareto (LBWP) and length-biased weighted Lomax (LBWL) distributions by taking $\lambda=1$, $\lambda=\alpha=1$ and $\alpha=1$ respectively.

Table 1: Sub-models of the LBEL distribution

No.	**Distribution**	α	θ	λ	**Author**
1	LBWEP	α	θ	1	New
2	LBWP	1	θ	1	New
3	LBWL	1	θ	λ	Afaq et al (2016)

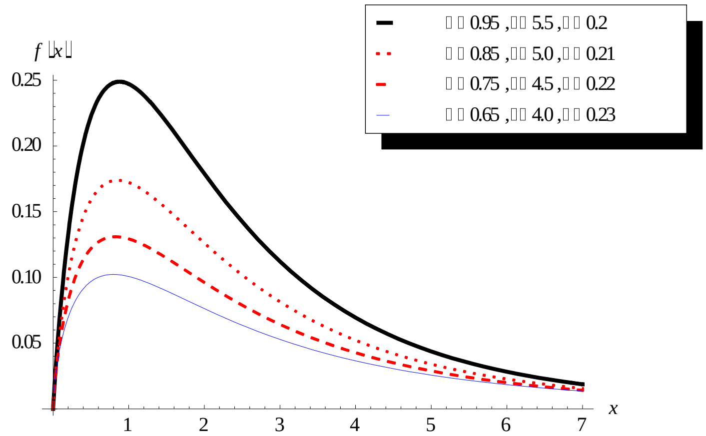

Figure 1 *pdf* of LBWELD under different values of parameters.

The cumulative distribution function $F(x)$, survival (reliability) function $S(x)$, the hazard rate function (HRF) $h(x)$ and the reversed hazard rate function (RHRF) $h^*(x)$ for LBWELD are in the following forms:

$$F(x)=\frac{\theta\lambda^2\int_0^x y\left[1-(1+\lambda y)^{-\theta}\right]^{\alpha-1}(1+\lambda y)^{-(\theta+1)}dy}{D_1}$$

Using substitution

$$\left\{\begin{array}{l} z=(1+\lambda y)^{-\theta} \\ \left|\frac{dz}{dy}\right|=\theta\lambda(1+\lambda y)^{-(\theta+1)} \end{array}\right\}. \qquad (5)$$

we get

$$F(x)=\frac{B\left(1-\frac{1}{\theta},\ \alpha\ ;(1+\lambda x)^{-\theta}\right)-\frac{1}{\alpha}\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha}}{D_1}, \qquad (6)$$

where $\beta(a,b;x)=\int_x^1 u^{a-1}(1-u)^{b-1}du$ is an upper incomplete beta function.

$$S(x)=\frac{D_1-B\left(1-\frac{1}{\theta},\alpha;(1+\lambda x)^{-\theta}\right)+\frac{1}{\alpha}\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha}}{D_1}, \qquad (7)$$

$$h(x)=\frac{\theta\lambda^{2}x\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha-1}(1+\lambda x)^{-(\theta+1)}}{D_{1}-B\left(1-\frac{1}{\theta},\alpha;(1+\lambda x)^{-\theta}\right)+\frac{1}{\alpha}\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha}}, \tag{8}$$

and

$$h^{*}(x)=\frac{\theta\lambda^{2}x\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha-1}(1+\lambda x)^{-(\theta+1)}}{B\left(1-\frac{1}{\theta},\ \alpha\ ;(1+\lambda x)^{-\theta}\right)-\frac{1}{\alpha}\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha}}. \tag{9}$$

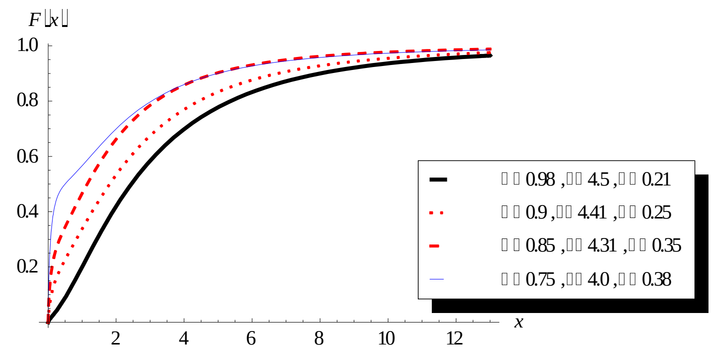

Figure 2 *CDF* of LBWELD under different values of parameters.

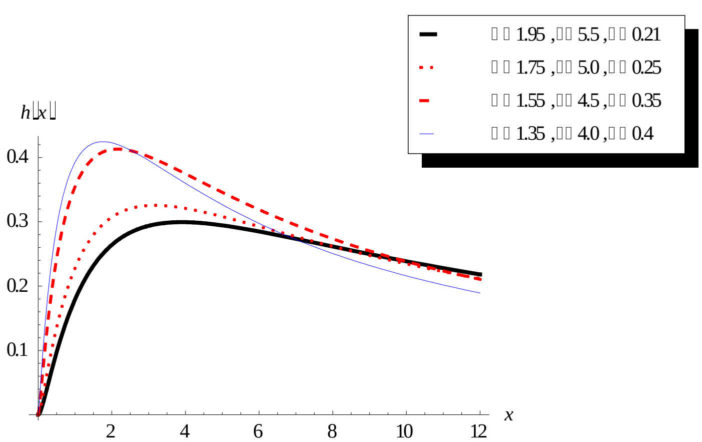

Figure 3 HRF of LBWELD under different values of parameters.

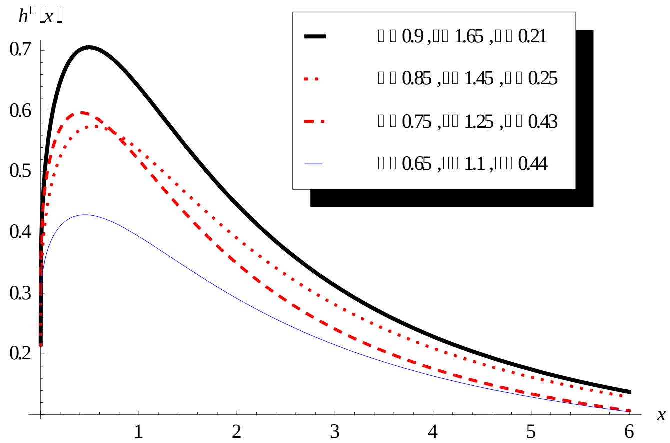

Figure 4 RHRF of LBWELD under different values of parameters.

3. Statistical Properties

In this section some statistical properties of Length-Biased weighted exponentiated Lomax distribution will be discuss.

3.1. Harmonic mean

The harmonic mean (H) of a random variable X with *pdf* $f(x)$ is given by the following formula

$$\frac{1}{H}=E\left(\frac{1}{X}\right) \tag{10}$$

Theorem

The harmonic mean of Length-Biased weighted of any distribution is equal to the mean of the base distribution.

Proof:

Suppose $f(x)$ is the *pdf* of Length-Biased weighted of any distribution, then

$$f(x)=\frac{xg(x)}{E(X)},\ x>0$$

where $g(x)$ is the *pdf* of base distribution with mean $E(X)$.

Using the formula (10), we get

$$\frac{1}{H}=E\left(\frac{1}{X}\right)=\int_0^\infty \frac{f(x)}{x}dx$$

$$=\frac{1}{E(X)}\int_0^\infty g(x)dx=\frac{1}{E(X)}$$

This is implies that

$$H = E(X) \qquad \blacksquare$$

Corollary:

The harmonic mean for LBWELD is

$$H = \frac{\alpha}{\lambda} D_1.$$

3.2. Moments

The r^{th} traditional moments for LBWELD is

$$\mu_r' = E\left(X^r\right) = \frac{\theta\lambda^2}{D_1}\int_0^\infty x^{r+1}\left(1+\lambda x\right)^{-(\theta+1)}\left[1-\left(1+\lambda x\right)^{-\theta}\right]^{\alpha-1} dx$$

Using substitution (5), we get

$$\mu_r' = \frac{\lambda^{-r}}{D_1}\int_0^1 \left(z^{-\frac{1}{\theta}}-1\right)^{r+1}\left(1-z\right)^{\alpha-1} dz$$

$$= \frac{\lambda^{-r}}{D_1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i \int_0^1 z^{-\frac{r+1-i}{\theta}}\left(1-z\right)^{\alpha-1} dz$$

$$= \frac{\lambda^{-r}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i \,\beta\left(1-\frac{r+1-i}{\theta},\alpha\right)}{D_1}; \quad r = 1,2,... \tag{11}$$

The first two moments can be obtained by taking r =1 and 2 in (11) as follows:

$$\mu_1' = \frac{\lambda^{-1}\sum_{i=0}^{2}\binom{2}{i}(-1)^i \,\beta\left(1-\frac{2-i}{\theta},\alpha\right)}{D_1} = \frac{D_2-D_1}{\lambda D_1}, \tag{12}$$

and

$$\mu_2' = \frac{\lambda^{-2}\sum_{i=0}^{3}\binom{3}{i}(-1)^i \,\beta\left(1-\frac{3-i}{\theta},\alpha\right)}{D_1} = \frac{D_3-2D_2+D_1}{\lambda^2 D_1}, \tag{13}$$

The variance (σ^2), standard deviation (σ) and coefficient of variation (CV) for LBWELD are

$$\sigma^2 = \frac{D_3 - 2D_2 + D_1}{\lambda^2 D_1} - \frac{\left(D_2 - D_1\right)^2}{\lambda^2 \left(D_1\right)^2} = \frac{D_1 D_3 - \left(D_2\right)^2}{\lambda^2 \left(D_1\right)^2} \tag{14}$$

$$\sigma = \frac{\sqrt{D_1 D_3 - \left(D_2\right)^2}}{\lambda D_1} \tag{15}$$

and

$$\text{CV} = \frac{\sqrt{D_1 D_3 - \left(D_2\right)^2}}{D_2 - D_1}. \tag{16}$$

Table 2: Mean, variance and coefficient of variation of LBWELD for selected values of the parameters.

Parameters			Mean	Variance	coefficient of variation
α	θ	λ			
1	4	0.5	2	8	1.414
2	5	1	0.786	0.638	1.016
3	6	1.5	0.437	0.136	0.843
4	7	2	0.282	0.044	0.742
5	8	2.5	0.2	0.018	0.674

3.3. Moment generating function

The moment generating function, $M\left(t\right)$, is given by

$$\begin{aligned} M\left(t\right) &= E\left(e^{tX}\right) = \int_0^\infty e^{tx} f\left(x\right)dx \\ &= \sum_{j=0}^{\infty} \frac{t^j}{j!} \int_0^\infty x^j f\left(x\right)dx \\ &= \sum_{j=0}^{\infty} \frac{t^j \mu_j'}{j!} \end{aligned}$$

This is implies that

$$M\left(t\right) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \frac{\lambda^{-j} \sum_{i=0}^{j+1} \binom{j+1}{i} \left(-1\right)^i \beta\left(1 - \frac{j+1-i}{\theta}, \alpha\right)}{D_1}$$

3.4. Mode

The mode of LBWELD is the solve the following equation with respect to x

$$1-\theta\lambda x+\left(1+\lambda x\right)^{-\theta}\left(\alpha\theta\lambda x-1\right)=0 \tag{17}$$

3.5 Information entropies

The Shannon and Reny entropy for LBWELD have been obtained in this section.

3.5.1 Shannon entropy

The Shannon entropy for any distribution can be defined as $E\left[-\ln f\left(x\right)\right]$.

For LBWELD the Shannon entropy is

$$\begin{aligned}E\left[-\ln f\left(x\right)\right]&=\ln\left(D_1\right)-\ln\left(\theta\right)-2\ln\left(\lambda\right)-E\left[\ln\left(X\right)\right]\\&-\left(\alpha-1\right)E\left[\ln\left[1-\left(1+\lambda X\right)^{-\theta}\right]\right]+\left(\theta+1\right)E\left[\ln\left(1+\lambda X\right)\right]\\&=\ln\left(D_1\right)-\ln\left(\theta\right)-2\ln\left(\lambda\right)-I_1-\left(\alpha-1\right)I_2+\left(\theta+1\right)I_3\end{aligned} \tag{18}$$

Where

$$\begin{aligned}I_1=E\left[\ln\left(X\right)\right]&=\frac{\theta\lambda^2}{D_1}\int_0^\infty x\ln\left(x\right)\left[1-\left(1+\lambda x\right)^{-\theta}\right]^{\alpha-1}\left(1+\lambda x\right)^{-(\theta+1)}dx\\&=\frac{\theta\lambda^2}{D_1}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}\left(-1\right)^i\int_0^\infty x\ln\left(x\right)\left(1+\lambda x\right)^{-\theta(i+1)-1}dx\end{aligned}$$

Integrating by parts we get

$$\begin{aligned}I_1&=\frac{\theta\lambda}{D_1}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}\frac{\left(-1\right)^i}{\theta\left(i+1\right)}\left\{\int_0^\infty\left(1+\lambda x\right)^{-\theta(i+1)}dx+\int_0^\infty\ln\left(x\right)\left(1+\lambda x\right)^{-\theta(i+1)}dx\right\}\\&=\frac{\theta\lambda}{D_1}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}\frac{\left(-1\right)^i}{\theta\left(i+1\right)}\left\{\frac{1}{\lambda\left[\theta\left(i+1\right)+1\right]}-\frac{\lambda}{\theta\left(i+1\right)-1}\left[\ln\left(\lambda\right)+C+\Psi\left(\theta\left(i+1\right)-1\right)\right]\right\},\end{aligned} \tag{19}$$

$$I_2 = E\left[\ln\left(1-\left(1+\lambda X\right)^{-\theta}\right)\right]$$
$$= \frac{\theta\lambda^2}{D_1}\int_0^\infty x\ \ln\left(1-\left(1+\lambda x\right)^{-\theta}\right)\left[1-\left(1+\lambda x\right)^{-\theta}\right]^{\alpha-1}\left(1+\lambda x\right)^{-(\theta+1)}dx$$
$$= \frac{\theta\lambda^2}{D_1}\sum_{i=0}^\infty\binom{\alpha-1}{i}(-1)^i\int_0^\infty x\ \ln\left(1-\left(1+\lambda x\right)^{-\theta}\right)\left(1+\lambda x\right)^{-\theta(i+1)-1}dx$$

Using substitution (5), we get

$$I_2 = \frac{1}{D_1}\sum_{i=0}^\infty\binom{\alpha-1}{i}(-1)^i\int_0^1 z^i\left(z^{-\frac{1}{\theta}}-1\right)\ln(1-z)\,dz$$
$$= \frac{\sum_{i=0}^\infty\binom{\alpha-1}{i}(-1)^i}{D_1}\left\{\frac{1}{i+1}\left[\Psi(i+2)+C\right]-\frac{1}{i-\frac{1}{\theta}+1}\left[\Psi\left(i-\frac{1}{\theta}+2\right)+C\right]\right\}, \qquad (20)$$

and

$$I_3 = E\left[\ln\left(1+\lambda X\right)\right]$$
$$= \frac{\theta\lambda^2}{D_1}\int_0^\infty x\ \ln\left(1+\lambda x\right)\left[1-\left(1+\lambda x\right)^{-\theta}\right]^{\alpha-1}\left(1+\lambda x\right)^{-(\theta+1)}dx$$
$$= \frac{\theta\lambda^2}{D_1}\sum_{i=0}^\infty\binom{\alpha-1}{i}(-1)^i\int_0^\infty x\ \ln\left(1+\lambda x\right)\left(1+\lambda x\right)^{-\theta(i+1)-1}dx$$

Using substitution $\left\{\begin{array}{l} z=(1+\lambda x)^{-1} \\ \left|\frac{dz}{dx}\right| = \lambda(1+\lambda x)^{-2}\end{array}\right\}$.

$$I_3 = \frac{-\theta}{D_1}\sum_{i=0}^\infty\binom{\alpha-1}{i}(-1)^i\int_0^1 z^{\theta(i+1)-1}\left(\frac{1}{z}-1\right)\ln(z)\,dz$$
$$= \frac{\theta\sum_{i=0}^\infty\binom{\alpha-1}{i}(-1)^i}{D_1}\left\{\frac{1}{\theta(i+1)-1}\left[\Psi\left(\theta(i+1)\right)+C\right]-\frac{1}{\theta(i+1)}\left[\Psi\left(\theta(i+1)+1\right)+C\right]\right\} \qquad (21)$$

Where $\Psi(x)=\frac{d}{dx}\ln\left(\Gamma(x)\right)$ and C is Eular constant.

Using the results (19), (20) and (21) in (18) and simplifying, we get the shannon entropy as:

$$E\left[-\ln f\left(x\right)\right]=\ln\left(D_1\right)-\ln\left(\theta\right)-2\ln\left(\lambda\right)-\frac{\sum_{i=0}^{\infty}\binom{\alpha-1}{i}\left(-1\right)^i}{D_1}\left\{\frac{1}{\left(i+1\right)\left[\theta\left(i+1\right)+1\right]}\right.$$

$$-\frac{\theta\left[\left(\theta+1\right)\Psi\left(\theta\left(i+1\right)\right)+\left(\alpha-1\right)\Psi\left(i-\frac{1}{\theta}+2\right)\right]}{\theta\left(i+1\right)-1}+ \qquad \mathbf{3.5.2}$$

$$\left.\frac{\left(\alpha-1\right)\Psi\left(i+2\right)+\left(\theta+1\right)\Psi\left(\theta\left(i+1\right)+1\right)}{i+1}-\frac{\lambda^2\left[C+\ln\left(\lambda\right)+\Psi\left(\theta\left(i+1\right)-1\right)\right]+\left(\alpha+\theta\right)C}{\left(i+1\right)\left[\theta\left(i+1\right)-1\right]}\right\}$$

Renyi entropy

Renyi entropy is defined as

$$I_R\left(\gamma\right)=\frac{1}{\gamma-1}\log\int_R f^{\gamma}\left(x\right)dx;\quad \gamma>0 \text{ and } \gamma\neq 1.$$

Now using the density function of LBWELD, we get

$$\int_R f^{\gamma}\left(x\right)dx=\frac{\theta^{\gamma}\lambda^{2\gamma}}{\left(D_1\right)^{\gamma}}\int_0^{\infty}x^{\gamma}\left[1-\left(1+\lambda x\right)^{-\theta}\right]^{\gamma\left(\alpha-1\right)}\left(1+\lambda x\right)^{-\gamma\left(\theta+1\right)}dx$$

Using substitution (5), we get

$$\int_R f^{\gamma}\left(x\right)dx=\frac{\theta^{\gamma-1}\lambda^{\gamma-1}}{\left(D_1\right)^{\gamma}}\int_0^1\left(z^{-\frac{1}{\theta}}-1\right)^{\gamma}\left(1-z\right)^{\gamma\left(\alpha-1\right)}z^{-\frac{\left(\theta+1\right)\left(1-\gamma\right)}{\theta}}dz$$

$$=\frac{\theta^{\gamma-1}\lambda^{\gamma-1}}{\left(D_1\right)^{\gamma}}\sum_{i=0}^{\infty}\binom{\gamma}{i}\left(-1\right)^i\int_0^1\left(1-z\right)^{\gamma\left(\alpha-1\right)}z^{-\frac{\left(\theta+1\right)\left(1-\gamma\right)+\gamma-i}{\theta}}dz$$

$$=\frac{\theta^{\gamma-1}\lambda^{\gamma-1}}{\left(D_1\right)^{\gamma}}\sum_{i=0}^{\infty}\binom{\gamma}{i}\left(-1\right)^i\beta\left(1-\frac{\left(\theta+1\right)\left(1-\gamma\right)+\gamma-i}{\theta},\gamma\left(\alpha-1\right)+1\right).$$

Then, we get the Renyi entropy as:

$$I_R\left(\gamma\right)=\frac{1}{\gamma-1}\log\left[\frac{\theta^{\gamma-1}\lambda^{\gamma-1}\sum_{i=0}^{\infty}\binom{\gamma}{i}\left(-1\right)^i}{\left(D_1\right)^{\gamma}}\beta\left(1-\frac{\left(\theta+1\right)\left(1-\gamma\right)+\gamma-i}{\theta},\gamma\left(\alpha-1\right)+1\right)\right].$$

4. Maximum Likelihood Estimators (MLE)

In this section, we consider maximum likelihood estimators (MLE) of LBWELD. Let $x_1, x_2, ..., x_n$ be a random sample of size n from LBWELD, then the log-likelihood function $L(\lambda,\theta,\alpha)$ can be written as

$$L(\lambda,\theta,\alpha) \propto n\left[2\ln(\lambda)+\ln(\theta)+\ln(\alpha)+\ln\left(\Gamma\left(\alpha+1-\frac{1}{\theta}\right)\right)\right]$$
$$+\sum_{i=1}^{n}\ln(x_i)+(\alpha-1)\sum_{i=1}^{n}\ln\left[1-(1+\lambda x_i)^{-\theta}\right]-(\theta+1)\sum_{i=1}^{n}\ln(1+\lambda x_i)$$
$$-n\ln\left[\Gamma\left(1-\frac{1}{\theta}\right)\Gamma(\alpha+1)-\Gamma\left(\alpha+1-\frac{1}{\theta}\right)\right] \qquad (22)$$

Then

$$\frac{\partial L}{\partial \alpha}=n\left[\frac{1}{\alpha}+\psi\left(\alpha+1-\frac{1}{\theta}\right)\right]+\sum_{i=1}^{n}\ln\left[1-(1+\lambda x_i)^{-\theta}\right]$$
$$-\frac{n\Gamma(\alpha+1)\Gamma\left(\alpha+1-\frac{1}{\theta}\right)\left[\frac{\Gamma\left(1-\frac{1}{\theta}\right)\psi(\alpha+1)}{\Gamma\left(\alpha+1-\frac{1}{\theta}\right)}+\frac{\psi\left(\alpha+1-\frac{1}{\theta}\right)}{\Gamma(\alpha+1)}\right]}{\Gamma\left(1-\frac{1}{\theta}\right)\Gamma(\alpha+1)-\Gamma\left(\alpha+1-\frac{1}{\theta}\right)} \qquad (23)$$

$$\frac{\partial L}{\partial \theta}=\frac{n}{\theta^2}\left[\theta+\psi\left(\alpha+1-\frac{1}{\theta}\right)\right]+(\alpha-1)\sum_{i=1}^{n}\frac{(1+\lambda x_i)^{-\theta}\ln(1+\lambda x_i)}{\left[1-(1+\lambda x_i)^{-\theta}\right]}-\sum_{i=1}^{n}\ln(1+\lambda x_i)$$
$$-\frac{n}{\theta^2}\frac{\Gamma\left(1-\frac{1}{\theta}\right)\Gamma\left(\alpha+1-\frac{1}{\theta}\right)\left[\frac{\Gamma(\alpha+1)\psi\left(1-\frac{1}{\theta}\right)}{\Gamma\left(\alpha+1-\frac{1}{\theta}\right)}-\frac{\psi\left(\alpha+1-\frac{1}{\theta}\right)}{\Gamma\left(1-\frac{1}{\theta}\right)}\right]}{\Gamma\left(1-\frac{1}{\theta}\right)\Gamma(\alpha+1)-\Gamma\left(\alpha+1-\frac{1}{\theta}\right)} \qquad (24)$$

$$\frac{\partial L}{\partial \lambda}=\frac{2n}{\lambda}+\theta(\alpha-1)\sum_{i=1}^{n}\frac{x_i(1+\lambda x_i)^{-(\theta+1)}}{\left[1-(1+\lambda x_i)^{-\theta}\right]}-(\theta+1)\sum_{i=1}^{n}\frac{x_i}{(1+\lambda x_i)} \qquad (25)$$

The MLE of λ, θ and α can be obtain by solving the equations (23), (24), and (25) using $\frac{\partial L}{\partial \lambda}=0$, $\frac{\partial L}{\partial \theta}=0$ and $\frac{\partial L}{\partial \alpha}=0$.

4.1 Asymptotic confidence bounds

We derive it for these parameters when α, θ > 0 and q > 0 as the MLEs of the unknown parameters α, θ and λ can't be obtained in closed forms, by using variance covariance matrix see (Lawless(2003)), where

$$I=\begin{bmatrix} -\frac{\partial^2 L}{\partial \lambda^2} & -\frac{\partial^2 L}{\partial \lambda \partial \theta} & -\frac{\partial^2 L}{\partial \lambda \partial \alpha} \\ -\frac{\partial^2 L}{\partial \theta \partial \lambda} & -\frac{\partial^2 L}{\partial \theta^2} & -\frac{\partial^2 L}{\partial \theta \partial \alpha} \\ -\frac{\partial^2 L}{\partial \alpha \partial \lambda} & -\frac{\partial^2 L}{\partial \alpha \partial \theta} & -\frac{\partial^2 L}{\partial \alpha^2} \end{bmatrix}^{-1}$$

Thus

$$I=\begin{bmatrix} \operatorname{var}\left(\hat{\lambda}\right) & \operatorname{cov}\left(\hat{\lambda},\hat{\theta}\right) & \operatorname{cov}\left(\hat{\lambda},\hat{\alpha}\right) \\ \operatorname{cov}\left(\hat{\theta},\hat{\lambda}\right) & \operatorname{var}\left(\hat{\theta}\right) & \operatorname{cov}\left(\hat{\theta},\hat{\alpha}\right) \\ \operatorname{cov}\left(\hat{\alpha},\hat{\lambda}\right) & \operatorname{cov}\left(\hat{\alpha},\hat{\theta}\right) & \operatorname{var}\left(\hat{\alpha}\right) \end{bmatrix}$$

So, we can get the $(1-\delta\ )100\%$ confidence intervals of the parameters α, θ and λ as $\hat{\alpha}\pm Z_{\frac{\delta}{2}}\sqrt{\operatorname{var}\left(\hat{\alpha}\right)}$, $\hat{\theta}\pm Z_{\frac{\delta}{2}}\sqrt{\operatorname{var}\left(\hat{\theta}\right)}$ and $\hat{\lambda}\pm Z_{\frac{\delta}{2}}\sqrt{\operatorname{var}\left(\hat{\lambda}\right)}$.

5. Applications

In this Section we fit LBWELD to two real data sets and compare the fitness with the exponentiated Lomax (EL) and length-biased weighted Lomax (LBWL) distributions, whose densities are given by

$$f_{LD}(x;\alpha,\theta,\lambda)=\alpha\theta\lambda\left[1-(1+\lambda\,x)^{-\theta}\right]^{\alpha-1}(1+\lambda\,x)^{-(\theta+1)};\quad x>0,\ \left(\alpha,\theta\, and\ \lambda>0\right),$$

$$f_{LBWL}(x;\ \theta,\lambda)=\frac{\theta(\theta-1)}{\lambda^2}x\,(1+\frac{x}{\lambda})^{-(\theta+1)};\ x>0,\ \left(\theta,\lambda>0\right).$$

Specifically, we consider two data sets. The first set of data represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). The second set represents the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. The data set is obtained from Smith and Naylor (1987).

In order to compare distributions, we consider the K-S (Kolmogorov-Smirnov) statistic, -2logL, AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected), BIC (Bayesian Information Criterion). The best distribution corresponds to lower -2logL, AIC, BIC, AICC statistics value.

Where,

$$AIC = 2m - 2\ln L, \quad AICC = AIC + \frac{2m(m+1)}{n-m-1},$$

$$BIC = m\ln(n) - 2\ln L \text{ and } K-S = \max_{1\le i\le n}\left(F(x_i) - \frac{i-1}{n}, \frac{i}{n} - F(x_i)\right)$$

where $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{x_i \le x}$ is empirical distribution function, $F(x)$ is comulative distribution function, m is the number of parameters in the statistical model, n the sample size and .

Table 3. Maximum-likelihood estimates, AIC, BIC and AICC values, and K-S statistics for the 72 guinea pigs infected with virulent tubercle bacilli.

Model	**MLEs**			**Measures**				
	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	**K-S**	**-2logL**	**AIC**	**BIC**	**AICC**
LBWELD	0.766	2.7	0.957	0. 82	188.796	194.796	201.626	200.854
EL	0.249	2.172	0.797	0.321	199.983	205.983	212.813	212.042
LBWL	0.324	1.796	--	0.333	260.247	264.247	268.8	268.291

The variance covariance matrix is

$$I = \begin{bmatrix} 4.906\times10^{-3} & -0.171 & 2.828\times10^{-3} \\ -0.171 & 6.398 & -0.094 \\ 2.828\times10^{-3} & -0.094 & 8.415\times10^{-3} \end{bmatrix}$$

The approximate 95% two sided confidence interval of the parameters λ, θ and α are [0.629, 0.903], [-2.258, 7.658] and [0.777, 1.137] respectively.

Table 4. Maximum-likelihood estimates, AIC, BIC and AICC values, and K-S statistics for the strengths of 1.5 cm glass fibres.

Model	**MLEs**			**Measures**				
	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	**K-S**	**-2logL**	**AIC**	**BIC**	**AICC**
LBWELD	0.377	4.745	1.113	0.431	139.809	145.809	152.191	151.878
EL	0.21	2.199	0.677	0.413	172.106	178.106	184.487	184.175
LBWL	0.668	2.373	--	0.419	187.671	191.671	195.925	195.772

The variance covariance matrix is

$$I = \begin{bmatrix} 3.864\times10^{-3} & -0.122 & 1.938\times10^{-3} \\ -0.122 & 4.37 & -0.058 \\ 1.938\times10^{-3} & -0.058 & 8.592\times10^{-3} \end{bmatrix}$$

The approximate 95% two sided confidence interval of the parameters λ, θ and α are [0.255, 0.499], [0.648, 8.842] and [0.931, 1.295] respectively.

Table 3 and Table 4 show parameter MLEs, the values of K_S, -2logL, AIC, BIC, AICC statistics for the three data set consecutively. From the above results, it is evident that the LBWELD distribution is the best distribution for fitting these data sets compared to other distributions considered here. And is a strong competitor to other distributions commonly used in literature for fitting lifetime data.

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