USING A NEW MATHEMATICAL TECHNIQUE TO FIND NORMAL CURVATURE ON SMOOTH SURFACES IN CYLINDRICAL SYSTEM

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Abstract:
We have dealt with some aspects of curvature and normal curvature. The purpose of this research is to find the normal curvature numerically in the cylindrical coordinate system using Matlab. Matlab is an interactive working environment that allows users to perform fairly complex computational tasks using only a few commands. We followed the Applied Mathematical method using a new mathematical technique (Matlab) and we found that finding the normal curvature in a cylindrical coordinate system using a new mathematical technique is more accurate and faster than numerical finding.

Keywords: *New Mathematical Technique, Matlab , Normal Curvature, Cylindrical System.*

1.INTRODUCTION:

The differential geometry of curves and surfaces has two aspects. One, which may be called classical differential geometry, started with the beginnings of calculus. Roughly speaking, classical differential geometry is the study of local properties of curves and surfaces. By local properties we mean those properties which depend only on the behavior of the curve or surface in then neighborhood of a point. The methods which have shown themselves to beadequate in the study of such properties are the methods of differential calculus. Because of this, the curves and surfaces considered in differential geome try will be defined by functions which can be differentiated a certain number of times. [13,pp1] Differential geometry is a branch of mathematics using calculus to study the geometric properties of curves and surfaces. It arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces. The theory developed in this study originates from mathematicians of the 18th and 19th centuries, mainly; Euler (1707-1783), Monge (1746-1818) and Gauss (1777-1855). Mathematical study of curves and surfaces has been developed to answer some of the nagging and unanswered questions that appeared in calculus, such as the reasons for relationships between complex shapes and curves, series and analytic functions11. Study of curvatures is an important part of differential geometry[11,pp6].The concept of Curvature and its related, constitute the central object of study in differential Geometry[8,pp51] Differential geometry is largely concerned with the same problemsas Euclidean geometry{namely how to measure lengths, angles, and areas{but done in a more general setting usingthe tools of calculus and linear algebra [4,pp1].The curvatures of a smooth surface are local measures of its shape. Here we consideranalogous quantities for discrete surfaces, meaning triangulated polyhedral surfaces. Often the most useful analogs are those which preserve integral relations for curvature, like the Gauss/Bonnet theorem or the force balance equation for mean curvature. For simplicity, we usually restrict our attention to surfaces in euclidean three-space E^3, although some of the results generalize to other ambient manifolds of arbitrary dimension.[10,pp1]

2.Curves:

A regular curve is a parametrized curve whose velocity is never zero. A regularcurve can be reparametrized by arc length.

i.Regular Curves:

Definition (2.1): A parametrized curve $c: [a,b] \to M$ is regular if its velocity $c'(t)$is never zero for all t in the domain $[a,b]$. In other words, a regular curve in M is an immersion: $[a,b] \to M$.[12,pp9]

Definition (2.2): A parametrized differentiable curve $\alpha: I \to R^3$ is said to be regular if $\alpha'(t) \neq 0$ for all $t \in I$..[13,pp22]

ii.Plane Curves:

Curves are one-dimensional geometric objects which arestraight or curved within a higher dimensional ambient space.They are widely used to represent thin physical objects such asrods and wires, as well as to describe the trails of movingobjects. Less common but no less interesting examples ofcurves include singular features under physical processes withconcentration mechanisms. e.g. rivers in an eroded terrain ortornadoes in a fluid**.**

Mathematically, one views curves

a. Explicitly - a curve is a family of points that can becontinuously parametrized by a single variable.

b. Implicitly - a curve in the plane R^2 is the level set$\{(x, y) \in R^2 \mid f(x, y) = 0\}$ of a continuous scalar function $f : R^2 \to R$; a curve in R^3 is the intersection of two level setsurfaces $f(x, y, z) = 0, g(x, y, z) = 0$.

These two representations are locally equivalent in genericcases. [1, pp11]

iii. Smooth curves:

It will be convenient to strengthen the differentiability condition: a curve γ: $(a,b) \to R^3$is smooth if it is infinitely differentiable; that is, the nth derivative$\gamma^{(n)}(t)$ exists forall $t \in (a,b)$ and $n \geq 1$, where

$$\gamma^{(0)} := \gamma \text{ and}$$

$$\gamma^{(n)}(t) := \lim_{h \to 0} \frac{\gamma^{n-1}(t+h) - \gamma^{n-1}(t)}{h} \text{for all } t \in (a, b).$$

In terms of coordinates, the curve is smooth if and only if each of its coordinate functions is infinitely differentiable. [6,pp16]

3.Curvature:

For a unit-speed smooth space curve γ the magnitude of its acceleration$|\gamma''(s)|$ is called its curvature at the time s. Ifγ is simple, thenwe can say that $|\gamma''(s)|$ is the curvature at the point p $= \gamma(s)$ withoutambiguity. The curvature is usually denoted by $k(s)$ or $k(s)_\gamma$ and inthe case of simple curves it might be also denoted by $k(p)$ or$k(p)_\gamma$.The curvature measures how fast the curve turns; if you drive along a plane curve, then curvature describes the position of your steeringwheel at the given point (note that it does not depend on your speed). In general, the term curvature is used for anything that measureshow much a geometric object deviates from being straight; for curves,it measures how fast it deviates from a straight line.[3,pp27-28]

Definition (3.1):

We are looking for a function that measures the angle of rotation for the unit normal vector, or equivalently, the unit tangent vector. In terms of the Gaussmap, the head of the unit normal vector always lies on the unit circle. Therefore,the derivative of the unit normal vector must always be tangent to the unit circle.This is a manifestation of the fact that the derivative of a vector function that has constant magnitude is always perpendicular to the original vector function. Twonotions point the way. First, over small distances, the arc of a circle near a pointon the circle and the tangent line through that point are very similar. Second, the length of an arc of the unit circle is equal to the corresponding angle measured inradians. Therefore, a derivative of the unit normal vector measures change alonga tangent to the unit circle (as in the Gauss map), this change is essentially the same as the change along the unit circle, which is equal to a change in the directionof the normal vector measured in radians. In other words, the conclusions of thelast section suggest that the curvature can be defined as the derivative of the unit normal vector with respect to arc length. It can also be defined as the derivative ofthe unit tangent vector with respect to arc length.

That is: $k(s) = \left\| \frac{dn}{ds} \right\| = \left\| \frac{dT}{ds} \right\|$.[7,pp6-7]

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Definition (3.2): Let S be a surface, $p \in S$ and let $N(p) \in \mathbb{R}^3$ be a vector orthogonal to T_pS. Given $v \in T_pS$ of length 1, orient the plane $H_v$by choosing$\{v,N(p)\}$ as positive basis. The normal curvature of S at p along v is the oriented curvature at p of the normal section of S at p along v (considered as a plane curve contained in Hv). [14, pp184]

Definition (3.3): If is a unit-speed curve with parameter $t$its curvature $k(t)$ at $\gamma(t)$is defined to be $\| \gamma''(t) \|$.[19,pp13]

Definition (3.4): If two curves$\gamma_1(t)$ and $\gamma_2(u)$ intersect at a point P so that their derivatives point in the same direction, then we say that the two curves have firstordercontactat P. In this case, taking P to be the origin, and the curves to be moving in the direction of the positive X-axis and taking the normal vector to be on the left-hand side so that it points in the direction of the positive Y-axis at P, the two curves $\gamma_1(t)$ and $\gamma_2(u)$ given above can each be written as graphs of functions $y = f(x)$ and $y = g(x)$ so that:

$$F(0) = g(0) = 0 ,\frac{df}{dx}(0) = \frac{dg}{dx}(0) = 0 .$$

Then, if we further have: $\frac{d^2f}{dx^2}(0) = \frac{d^2g}{dx^2}(0)$ we say that the two curves have second or dercontact.pp16

Theorem (3.5): The osculating circle at a point $\gamma(t_0)$ of a regular curve $\gamma(t)$ makes second order contact with the curve at that point. Conversely, any circle that makes second order contact with $\gamma(t)$ at $\gamma(t_0)$ must be the osculating circle at that point.

Proof: Take a coordinate system (x, y) on the plane as in Definition (3.4) at $\gamma(t_0)$ and then the curve can be represented as a graph $y = f(x)$, where $f(0) = \frac{df}{dx}(0) = 0$. So the Taylor expansion of $f(x)$ at $x = 0$ is :

$$f(x) = \frac{k(t_0)}{2}x^2 + o(x^2). \qquad (3.1)$$

Here $o(.)$ is Landau's symbol.

The graph $y = f(x)$ has second order contact with the X-axis (i.e., the graph of $y = 0$) at the origin if and only if $k(t_0) = 0$. So, we consider only the case $k(t_0) \neq 0$ from now on. By reflecting the curve across the X-axis if necessary, we may assume $k(t_0) > 0$ without loss of generality. Here, the circle of radius a turning to the left and tangent to the X-axis at the origin can be represented as a graph $y = a - \sqrt{a^2 + x^2}$ which can be expanded as

$$y = \frac{1}{2a}x^2 + o(x^2) \qquad (3.2)$$

by (3.1) and the fact that the curvature of the circle is$\frac{1}{a}$. Comparing (3.1) and (3.2), we can conclude that the curve and the circle have second order contact at the origin if and only if $a = \frac{1}{k(t_0)}$ [16,pp16-17]

Theorem (3.6): The curvature of a regular curve, $\gamma(t)$. is given by :

$k(t) = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3}$ where a dash indicates a derivative wrt$t$. [16, pp15]

Proposition (3.7): A space curve is a line if and only if its curvature is everywhere 0.

Proof. The general line is given by $\alpha(s) = sv + c$for some unit vector $v$and constant vector c. Then$\alpha'(s) = T(s) = v$is constant, so $k = 0$. Conversely if $k = 0$ then $T(s)=T_0$is a constant vector and integrating, we obtain $\alpha(s) = \int_0^s T(u)du + \alpha(0) = s\,T_0 + \alpha(0)$. This is, once again, the parametricequation of a line.[17,pp15]

Definition (3.8): The Lie bracket $[X, Y]$ of two vector fields X and Yon a surface M is defined as the commutate or

$$[X, Y] = XY - YX ,$$

meaning that if f is a function on M, then $[X, Y](f) = X(Y(f)) - Y(X(f))$.

Proposition (3.9): The Lie bracket of two vectors X, $Y \in T(M)$ is another vector in $T(M)$.
Proof: If suffices to prove that the bracket is a linear derivation on the space of C^∞ functions.
Consider vectors $X, Y \in T(M)$ and smooth functions f, g in M. Then,

$$[X, Y](f + g) = X(Y(f + g)) - Y(X(f + g))$$
$$= X(Y(f)) - Y(X(f)) + X(Y(g)) - Y(X(g))$$
$$= [X, Y](f) + [X, Y](g),$$

and

$$[X, Y](fg) = X(Y(fg)) - Y(X(fg))$$
$$= X[fY(g) + gY(f)] - Y[fX(g) + gX(f)]$$
$$= X(f)Y(g) + fX(Y(g)) + X(g)Y(f) + gX(Y(f))$$
$$-Y(f)X(g) - f(Y(X(g)) - Y(g)X(f) - gY(X(f))$$
$$= f[X(Y(g)) - (Y(X(g))] + g[X(Y(f)) - Y(X(f))]$$
$$= f[X, Y](g) + g[X, Y](f).$$ [8,pp52]

i. Curvatures of Smooth Surfaces:
Given a (two-dimensional, oriented) surface M (smoothly immersed) in E^3we understand its local shape by looking at the Gauß map$v : M \to S^2$ given by the unit normal vector $v = v_p$ at each point$p \in M$. Its derivative at p is a linear map from $T_p M$to $T_{p_v}S^2$. Since these spaces are naturally identified, being parallel planesin E^3 we can view the derivative as an endomorphismS_p:$T_p M \to T_p M$. The map $S_p$is called the shape operator (orWeingarten map). The shape operator is the complete second-order invariant (orcurvature) which determines the original surface M. Usually however, it is more convenient not to work with the operatorS_p butinstead with scalar quantities. Its eigenvaluesk_1 and k_2 arecalled principal curvatures, and (since they cannot be globallydistinguished) it is their symmetric functions which have the most geometric meaning We define the Gauss curvature $K = k_1 k_2$as the determinant of $S_p$and the mean curvature $H = k_1 + k_2$ as its trace. Note that the sign of H depends on the choice of unit normal$v$. so often it is morenatural to work with the vector mean curvature (or mean curvaturevector) $H = Hv$ Note furthermore that some authors use the opposite sign on S_p and thus $H$and many us $H = \frac{k_1 + k_2}{2}$ justifying the name mean curvature. Our conventions mean that the mean curvature vector for a convex surface point inwards (like the curvature vector for a circle). For a unit sphere oriented with inward normal the Gauss map v is the antipodal map $S_p = I$ and $H \equiv 2$. [10,pp3].

ii. The Curvature of a Planar Smooth Curve:
A typical example of a geometric property of order 2 is the curvature $k(p)$ ofa (regular) curve C at a point p. To compute it, we first need to calculate theunit tangent vector field t, which involves a first derivative, and then take the derivative of the result. Using the arc length$s$, $(\frac{dt}{ds})_p = k(p)n(p)$, where n is the normal vector field of C. If the geometric image of the curve C is smooth enough, it can be locally represented by the graph $(x, f(x))$ of a smooth function f, such that $p = (0, 0)$, (*i.e.* $f(0) = 0$), and such that the tangent to C at p is collinear to the x-axis, (*i.e* $f'(0) = 0$). Then,
$k(p) = f''(0)$. Let $v = f'''(0)$. Near $p = (0, 0)$ we have:

$$f(x) = \frac{k\,x^2}{2} + \frac{v\,x^3}{6} + o(x^3).$$ [9,pp159-160]

4. Normal Curvature:
Definition (4.1): Let Sbe a regular surface of class C^2, and let $\vec{X} : U \to R^3$ be a parametrization of a coordinate neighborhood V of S.Let $\vec{\gamma} : I \to R^3$ be a parametrization of class C^2 for a curve C thatlies on S in V. The normal curvature of S along C is the function

$$k_n(t) = \frac{1}{s'}\vec{T'} \cdot \vec{N} = k(\vec{P}, \vec{N}) = k\cos\theta,$$

where$\theta$is the angle between the principal normal vector $\vec{N}$of thecurve and the normal vector $\sim N$ of the surface. [18,pp210]

Definition (4.2): Let S be a surface, $p \in S$ and let $N(p) \in R^3$ be a verso orthogonal to T_pS. Given $v \in T_pS$ of length 1, orient the plane $H_v$by choosing$\{v, N(p)\}$ as positive basis. The normal curvature of S at p along v is the oriented curvature at p of the normal section of S at p along v (considereredas a plane curve contained in H_v). [14,pp184]

Definition (4.3): Normal curvature depends only on the tangent vector t of the curve at P and not on the curve itself. For a non-zero tangent vector $t = ar_u + br_v$, and the definition of the normal curvature of S in direction t,

$$k_n = \frac{1}{|t|^2}((r_u \cdot v_u)\ a^2 + (r_u \cdot v_v + r_v \cdot v_u)ab + (r_v \cdot v_v)\ b^2).$$ [11, pp4]

Definition (4.4): Let p be a point of$M \subset \boldsymbol{R}^3$. The maximum and minimum values of the normal curvature $k(\boldsymbol{u})$ of Mat $p$are called the principal curvatures of M at$p$ and are denoted by k_1 and k_2. The directions in which these extreme values occur are called principal directions of M at$p$. Unit vectors in these directions are called principal vectors of M at p.

Definition (4.5): A point **p** of $M \subset \boldsymbol{R}^3$ is umbilic provided the normal curvature $k(\mathbf{u})$ is constant on all unit tangent vectors ***u*** at ***p***.[5,pp212]

Propositions (4.6): If γ is a unit-speed curve on an oriented surface *S*, its normal curvature is given by $k_n = \langle \gamma', \gamma' \rangle$. If σ is a surface patch of *S* and $\gamma(t) = \sigma(u(t), v(t))$ is a curve in σ,

$$k_n = L\,u'^2 + 2\,M\,u'v' + N v'^2$$

This result means that two curves which *touch* each other at a point ***p*** of a surface (i.e., which intersect at ***p*** and have parallel tangent vectors at ***p***) have the same normal curvature at ***p***.

Proof: Since γ' is a tangent vector to *S*, $\boldsymbol{N}.\gamma' = 0$. Hence, $\boldsymbol{N}.\gamma'' = -N'.\gamma''$ so

$$k_n = N.\gamma'' = N'.\gamma' = \langle W(\gamma'), \gamma' \rangle = \langle \gamma', \gamma' \rangle$$

since $N' = \frac{d}{dt} G(\gamma(t)) = -W(\gamma')$.[2,pp167]

Example (4.7):
Let M: $x^2 + y^2 = 1$ be a cylinder with $p=(1,0,0)$ and $V(p)=(1,0,0)$. A unit vector $u \in T_p(M)$ has the form $u = (0, u^1, u^2)$ with $(u^1)^2+(u^2)^2 = 1$. A normal for the plane determined by u and $U(p)$ is $(0, -u^2, u^1)$, so the plane's equation is $z = (u^2/u^1)y$. The intersection of the plane with M is the set $\{(\sqrt{1-y^2}, y, (u^2/u^1)y)\}$ for any y.
Parameterize σ by : $\sigma(t) = (\sqrt{1-t^2}, t, (\frac{u^2}{u^1})\,t)$
With
$\sigma'(t) = (-t/\sqrt{1-t^2}, 1, u^2/u^1)$ and $\sigma''(t) = (-1/(1-t^2)^{3/2}, 0, 0)$.

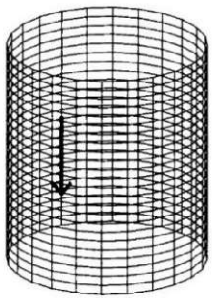
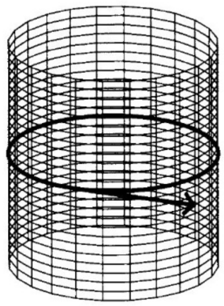

Figure (1) : Max; $k(u) = 0$; Figure (2) : Min; $k(u) = -I$;
$u^1 = 0$; u = (0,0, I) in direction $u^1 = 1$; u = (0, 1.0) in direction of rulings of velocity vector of directrix

Then we have,
$T(0) = (0, 1, u^2/u^1) / \sqrt{1 + (u^2/u^1)^2}$
$B(0) = \sigma'(0) \times \sigma''(0) / |\sigma'(0) \times \sigma''(0)|$
Hence , N(0) = - U(p), so we need a - sign in $k(u) = -k_\sigma(0)$. Further,

$$k_\sigma(0) = \frac{|\sigma'(0) \times \sigma''(0)|}{|\sigma'(0)|^3} = \frac{((u^2/u^1)^2+1)^{1/2}}{(1+(u^2/u^1)^2)^{3/2}} = (u^1)^2$$

Since $(u^1)^2 + (u^2)^2 = 1$. Hence, $k(u) = -(u^1)^2$. This is negative or zero. Now, since $u = (0, u^1, u^2)$ is on the unit circle in the *YZ*-plane, max $k(u) = 0$ occurs when $u^1 = 0$ and min $k(u) = -1$ occurs when $u^1 = 1$. The corresponding geometry is clear. The cylinder *M* is flat in the ruling directions and bends away from the normal in directrix directions. Indeed, the bending is what we might call circular.

Solution:

```
% Figure A
clearall
clc
symsrtu1u2sq
t = 0 : 0.2 : 1;
u1=0.2;
u2=0.2;
q=sqrt(1-t);
r=t;
s=(u1/u2)*t;
q=diff(q)
r=diff(r)
s=diff(s)
```

```
holdon
[q, r, s] =cylinder(q);
[q, r, s] =cylinder(r);
[q, r, s] =cylinder(s);
holdoff
surf(q, r, s);
gridon
```

Result:

```
q =
   -0.1056   -0.1198   -0.1421   -0.1852   -0.4472

r =
    0.2000    0.2000    0.2000    0.2000    0.2000

s =
    0.2000    0.2000    0.2000    0.2000    0.2000
```

Represent the Solution Graphically:

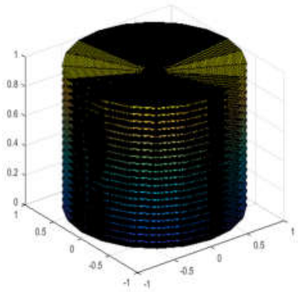

Figure (3): Max; $k(u) = 0; u^l = 0$; u = (0,0, I) in directionof rulings

Solution:

```
% Figure B
clearall
clc
symsrtu1u2sq
t = 0 : 0.2 : 1;
u1=0.2;
u2=0.2;
q=sqrt(1-t);
r=t;
s=(u1/u2)*t;
q=diff(diff(q))
r=diff(diff(r))
s=diff(diff(s))
holdon
[q, r, s] =cylinder(q);
[q, r, s] =cylinder(r);
[q, r, s] =cylinder(s);
holdoff
surf (q, r, s).
gridon
```

Result:

```
q =
   -0.0143   -0.0223   -0.0431   -0.2620

r =
   1.0e-15 *
        0   -0.0555    0.1110   -0.1110

s =
   1.0e-15 *
        0   -0.0555    0.1110   -0.1110
```

Represent the Solution Graphically:

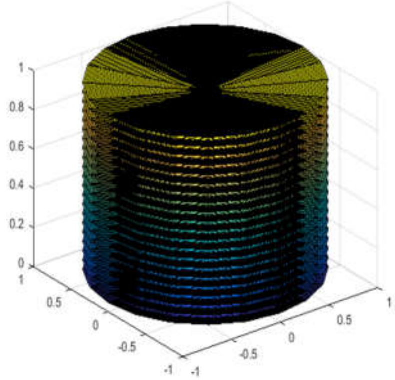

Figure (4): Min; *k(u)* = -I;u^1= 1 ; u = (0, 1.0) in Direction of Velocity Vector of Directrix

Results:

After we found the normal curvature in cylindrical system using a new mathematical technique we reached to the following some results :A new mathematical technique gives us a precise results of high speed compared with that of numerical, also we stated the ability capability of graphs or diagram drawing to any normal curvature via a new mathematical technique , we explained the possibility of the finding normal curvature by a new mathematical technique with a very high rate and accuracy finally we can considered a new mathematical technique as a theory which is considered one of the most important mathematical technique find the normal curvature and the other mathematical conceptions .

Conclusion:

Finally we can say that the method which we used in this paper help us to find the most accurate results and drawing them in a more rapid, attractive and clear way. Therefore, we hope that researchers will use this method (A New Mathematical Technique NMT) in their future scientific papers.

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