GENERALIZED ESTABLISH JENSEN TYPE ADDITIVE (λ_1, λ_2)-FUNCTIONAL INEQUALITIES WITH 3k-VARIABLES IN (α_1, α_2)-HOMOGENEOUS F-SPACES

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Abstract

In this paper, we study to solve two additives (λ_1, λ_2)-functional inequalities with 3k-variables in (α_1, α_2)-homogeneousF spaces. Then we will show that the solutions of the first and second inequalities are additive mappings. That is the main result in this paper.

Keywords: *Complex Banach space, Hyers-Ulam-Rassias stability, Additive (λ_1, λ_2)-Functional Inequalities, (α_1,α_2)-Homogeneous F spaces.*

Mathematics Subject Classifcation: *Primary 4610, 4710, 39B62, 39B72*,

1. INTRODUCTION

Let X and Y be a normed spaces on the same field K, and f : X → Y. We use the notation $\|\cdot\|$ for all the norm on both X and Y. In this paper, we investisgate some additive (λ_1, λ_2)-functional inequality in (α_1, α_2)-homogeneous F-spaces. In fact, when X is a α_1-homogeneous F-spaces and that Y is a α_2-homogeneous F-spaces we solve and prove the complex Banach space of two forllowing additive (λ_1, λ_2)-functional inequality.

$$\begin{aligned}&\Big\|f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j)+f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j-\sum_{j=1}^{k}z_j)-2\sum_{j=1}^{k}f(x_j)-2\sum_{j=1}^{k}f(y_j)\Big\|_{\mathbf{Y}}\\&\le\Big\|\lambda_1\Big(f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j)-\sum_{j=1}^{k}f(x_j)-\sum_{j=1}^{k}f(y_j)-\sum_{j=1}^{k}f(z_j)\Big)\Big\|_{\mathbf{Y}}\\&+\Big\|\lambda_2\Big(f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j-\sum_{j=1}^{k}z_j)-\sum_{j=1}^{k}f(x_j)-\sum_{j=1}^{k}f(y_j)+\sum_{j=1}^{k}f(z_j)\Big)\Big\|_{\mathbf{Y}}\qquad(1.1)\end{aligned}$$

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and when we change the role of the function inequality (1.1), we continue to prove the following function inequality.

$$\begin{aligned}&\Big\|f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j)-\sum_{j=1}^{k}f(x_j)-\sum_{j=1}^{k}f(y_j)-\sum_{j=1}^{k}f(z_j)\Big\|\\&\le\Big\|\lambda_1\Big(f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j)+\sum_{j=1}^{k}f(x_j-y_j)-\sum_{j=1}^{k}f(z_j)-2\sum_{j=1}^{k}f(x_j)\Big)\Big\|\\&+\Big\|\lambda_2\Big(f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j)-\sum_{j=1}^{k}f(x_j+y_j)-\sum_{j=1}^{k}f(z_j)\Big)\Big\|\qquad(1.2)\end{aligned}$$

where λ_1, λ_2 are fixed nonzero complex numbers with $G(\lambda_1, \lambda_2)$-functionalinequality. $\alpha_1, \alpha_2 \in \mathbb{R}+$, $\alpha_1, \alpha_2 \le 1$.

$$\Big(\mathbb{C}\setminus\{0\},\mathbf{Y}\Big)=\{G:\mathbb{C}\setminus\{0\}\to\mathbf{Y},G(\lambda_1,\lambda_2)=1-|\lambda_1|^{\alpha_2}-|\lambda_2|^{\alpha_2}<1\}$$

The Hyers-Ulam stability was first investigated for functional equation of Ulam in [1] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping.

The Hyers [2] gave firts affirmative partial answer to the equation of Ulam in Banach spaces. After that, Hyers'Theorem was generalized by Aoki [3] additive mappings and by Rassias [4] for linear mappings considering an unbouned Cauchy diffrence. Ageneralization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The Hyers-Ulam stability for functional inequalities has been investigated such as in [5],[6],[7]. Gilányi showed that if it satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\|\le\|f(x+y)\|\qquad(1.3)$$

Then f satisfies the Jordan-von Newman functional equation

$$2f(x)+2f(y)=f(x+y)+f(x-y)\qquad(1.4)$$

. Gilányi [6] and Fechner [8] proved the Hyers-Ulam stability of the functional inequality (1.3).

Next Choonkil Park [9] proved the Hyers-Ulam stability of additive β-functional inequal- ities. Recently, the author has studied the addition inequalities of mathematicians in the world as [5] [8] [10] -[24] and I have introduced two general additive function inequalities (1.1) and (1.2) based on the the additive function inequalities and the following additive functional equations

$$\Big\|\sum_{j=1}^{n}f\big(x_j\big)+\sum_{j=1}^{n}f\Big(\frac{x_{n+j}}{n}\Big)\Big\|\le\Big\|kf\Big(\frac{\sum_{j=1}^{n}x_j}{k}+\frac{\sum_{j=1}^{n}x_{n+j}}{n\cdot k}\Big)\Big\|,|n|>|k|.\qquad(1.5)$$

Next

$$\left\|f(x_1+x_2+...+x_n)-f(x_1)-f(x_2+...+x_n)\right\|_{\mathbf{Y}}$$
$$\leq \left\|\beta_1\Big(f(x_1+x_2+...+x_n)-f(x_1-x_2-...-x_n)-2f(x_1)\Big)\right\|_{\mathbf{Y}}$$
$$+\left\|\beta_2\Big(2f\Big(\frac{x_1+x_2+...+x_n}{2}\Big)-f(x_1)-f(x_2+...+x_n)\Big)\right\|_{\mathbf{Y}} \quad (1.6)$$

Next

$$\left\|\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+\sum_{j=1}^{k}f(z_j)\right\|_{\mathbf{Y}} \leq \left\|2kf\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j}{2k}\right)\right\|_{\mathbf{Y}}, \quad (1.7)$$

And

$$\left\|\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+\sum_{j=1}^{k}f(z_j)\right\|_{\mathbf{Y}} \leq \left\|f\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}, \quad (1.8)$$

And

$$\left\|\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+2k\sum_{j=1}^{k}f(z_j)\right\|_{\mathbf{Y}} \leq \left\|2kf\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j}{2k}+\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}. \quad (1.9)$$

Final

$$f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j)+f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j-\sum_{j=1}^{k}z_j)-2\sum_{j=1}^{k}f(x_j)-2\sum_{j=1}^{k}f(y_j)=0 \quad (1.10)$$

And

$$f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j)-f(\sum_{j=1}^{k}x_j-\sum_{j=1}^{k}y_j-\sum_{j=1}^{k}z_j)-2\sum_{j=1}^{k}f(y_j)-2\sum_{j=1}^{k}f(z_j)=0 \quad (1.11)$$

in Non-Archimedean Banach spaces and on the complex Banach space. When proving the additive function inequalities and the additive function equations on the complex Banach space, I continue to study the above additive (λ_1, λ_2) - function inequality on the (α_1, α_2)-homogeneous F-spaces. i.e., the a-functional inequalities with 3k-variables. Under suitable assumptions on spaces **X** and **Y**

, we will prove that the mappings satisfying the (λ_1, λ_2)-functional inequatilies (1.1) or (1.2). Thus, the results in this paper are generalization of those in [7] [9] [17] [25] [26] [27] for a-functional inequatilies with 3k-variables. The paper is organized as followns: In section preliminarier we remind a basic property such as We only redefine the solu- tion definition of the equation of the additive function. In this paper, I construct the additive Jensen (λ_1, λ_2)-function inequality on the (α_1, α_2)-homogeneous F-spaces with an unlimited number of variables to facilitate the construction of functional equations on the infinite-dimensional space. The method is that I rely on the ideas of mathematicians around the world See ([1]-[28]). This is a bright horizon for the function inequality. The paper is organized as followns: In section preliminarier we remind a basic property such as We only redefine the solution definition of the equation of the additive function and F^*-space .

Section 3: Establishing the solution for (1.1) in (α_1, α_2)-homogeneous F-spaces.
Section 4: Establishing the solution for (1.2) in (α_1, α_2)-homogeneous F-spaces.

2. Preliminarier

1. *F* *- **spaces.**

Definition 2.1.
Let **X** be a complex linear space. A nonnegative valued function $\|\cdot\|$ is an F -norm if it satisfies the following conditions:

(1) $\|x\| = 0$ if and only if $x = 0$;

(2) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all λ with $|\lambda| = 1$;

(3) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;

(4) $\|\lambda_n x\| \to 0,\ \lambda_n \to 0$;

(5) $\|\lambda x_n\| \to 0,\ x_n \to 0$.

(6) $\|\lambda_n x_n\| \to 0, \lambda_n \to 0,\ x_n \to 0$.

geneous ($\beta > 0$) if $\|tx\| = |t|^{\beta}\|x\|$ for all $x \in$ **X** and for all $t \in$ C and $(X, \|\cdot\|)$ is called α-homogeneous F-space

2.2 Solutions of the inequalities. The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the cauchuy equation. In particular, every solution of the cauchuy equation is said to be an *additive mapping*.

3. Establishing the solution for (1.1) in (α_1, α_2)-homogeneous *F* -spaces

3.1. Condition for existence of solutions for Equation (1.1). Here pay attention that **X** is a α_1-homogeneous F -spaces and that **Y** is a α_2-homogeneous F -spaces.

Lemma 3.1. *If a mapping* $f : \mathbf{X} \to \mathbf{Y}$ *sattisfies*

$$\begin{aligned} &\left\| f\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\Big) + f\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j - \sum_{j=1}^{k} z_j\Big) - 2\sum_{j=1}^{k} f(x_j) - 2\sum_{j=1}^{k} f(y_j) \right\|_{\mathbf{Y}} \\ &\leq \left\| \lambda_1 \Big(f\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k} f(x_j) - \sum_{j=1}^{k} f(y_j) - \sum_{j=1}^{k} f(z_j)\Big) \right\|_{\mathbf{Y}} \\ &+ \left\| \lambda_2 \Big(f\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j - \sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k} f(x_j) - \sum_{j=1}^{k} f(y_j) + \sum_{j=1}^{k} f(z_j)\Big) \right\|_{\mathbf{Y}} \qquad (3.1) \end{aligned}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ *for* $j = 1 \to n$, *then* $f: \mathbf{X} \to \mathbf{Y}$ *is additive*

Proof. Assume that $f : \mathbf{X} \to \mathbf{Y}$ satisfies (3 .1).

We replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by $(0, ..., 0, 0, ..., 0, 0, ..., 0)$ in (3.1), we have

$$\left\|(4k-2)f(0)\right\| \leq \left\|\lambda_1(3k-1)f(0)\right\| + \left\|\lambda_2(k-1)f(0)\right\|$$

Therefore

So $f(0) = 0$

Replacing $(x_1, ..., x_k, y_1, y_2, ..., y_k, z_1, ..., z_k)$ by $(0, ..., 0, 0, 0, ..., 0, z, ..., 0)$we get

$$\left\|f(y) + f(-y)\right\| \leq 0$$

and so f is an odd mapping. Replacing ($x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k$) by ($x_1, \ldots, x_k, 0, \ldots, 0, z_1, \ldots, z_k$) in (3.1), we have

$$\begin{aligned}&\left\|f\Big(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} z_j\Big)-\sum_{j=1}^{k} f(z_j)-\sum_{j=1}^{k} f(x_j)\right\|\\ &\leq\left\|\lambda_1\Big(f\Big(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} z_j\Big)-\sum_{j=1}^{k} f(x_j)-\sum_{j=1}^{k} f(z_j)\Big)\right\|\\ &+\left\|\lambda_2\Big(f\Big(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} z_j\Big)-\sum_{j=1}^{k} f(x_j)-\sum_{j=1}^{k} f(z_j)\Big)\right\| \qquad (3.2)\end{aligned}$$

And so

$$\left(1-\left|\lambda_1\right|^{\alpha_2}-\left|\lambda_2\right|^{\alpha_2}\right)\left\|f\Big(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} z_j\Big)-\sum_{j=1}^{k} f(z_j)-\sum_{j=1}^{k} f(x_j)\right\| \leq 0 \qquad (3.3)$$

And so

$$f\Big(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} z_j\Big)=\sum_{j=1}^{k} f(x_j)+\sum_{j=1}^{k} f(z_j)$$

for all $x_j, z_j \in \mathbf{X}$ for $j = 1 \rightarrow k$, as we expected. Q

3.2. Constructing a solution for (1.1). Now, we first study the solutions of (1.1). Note that for these inequalities, when **X** is a α_1-homogeneous F -spaces and that **Y** is a α_2-homogeneous F -spaces. Under this setting, we can show that the mapping satisfying (1.1) is additive. These results are give in the following.

Theorem 3.2. *suppose* $r > \frac{\alpha_2}{\alpha_1}, \theta$ *be nonngative real number, and let* $f: \mathbf{X} \rightarrow \mathbf{Y}$ *be a mapping such that*

$$\begin{aligned}&\left\|f\Big(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j\Big)+\sum_{j=1}^{k} f(x_j-y_j)-\sum_{j=1}^{k} f(z_j)-2\sum_{j=1}^{k} f(x_j)\right\|\\ &\leq\left\|\lambda_1\Big(f\Big(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j\Big)-\sum_{j=1}^{k} f(x_j)-\sum_{j=1}^{k} f(y_j)-\sum_{j=1}^{k} f(z_j)\Big)\right\|\\ &+\left\|\lambda_2\Big(f\Big(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j\Big)-\sum_{j=1}^{k} f(x_j+y_j)-\sum_{j=1}^{k} f(z_j)\Big)\right\|\\ &+\theta\Big(\sum_{j=1}^{k}\|x_j\|^r+\sum_{j=1}^{k}\|y_j\|^r+\sum_{j=1}^{k}\|z_j\|^r\Big) \qquad (3.4)\end{aligned}$$

for all xj, yj, zj ∈ X for all j = 1 → n. Then there exists a unique mapping ψ: X → Y such that

$$\left\|f(x)-\psi(x)\right\| \leq \frac{2k}{(1-|\lambda_1|^{\alpha_2}-|\lambda_2|^{\alpha_2})((2k)^{\alpha_1 r}-(2k)^{\alpha_2})}\theta\|x\|^r. \qquad (3.5)$$

for all $x \in$ x
for all x or all x ∈ X Proof. Assume that f: X → Y satisfies (3.4).
Replacing ($x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k$) by (0,..., 0, 0, ..., 0, 0, ..., 0) in (3.4), we have
So $f(0) = 0$

Next we replacing ($x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k$) by($x, \ldots, x, 0, \ldots, 0, x, \ldots, x$) in (3.4), we get

$$\left\|f(2kx)-2kf(x)\right\| \leq \frac{2k\theta}{(1-|\lambda_1|^{\alpha_2}-|\lambda_2|^{\alpha_2})}\left\|x\right\|^r \qquad (3.6)$$

for all x ∈ X. Thus

$$\left\|f(x)-2kf\Big(\frac{x}{2k}\Big)\right\| \leq \frac{2k\theta}{|2k|^{\alpha_1 r}(1-|\lambda_1|^{\alpha_2}-|\lambda_2|^{\alpha_2})}\left\|x\right\|^r \qquad (3.7)$$

for all x ∈ X. So

$$\Big\|(2k)^l f\big(\frac{x}{(2k)^l}\big) - (2k)^p f\big(\frac{x}{(2k)^p}\big)\Big\| \le \sum_{j=l}^{p-1} \Big\|(2k)^j f\big(\frac{x}{(2k)^j}\big) - (2k)^{j+1} f\big(\frac{x}{(2k)^{j+1}}\big)\Big\|$$

$$\le \frac{2k\theta}{|2k|^{\alpha_1 r}(1-|\lambda_1|^{\alpha_2}-|\lambda_2|^{\alpha_2})}\sum_{j=l}^{p-1}\frac{(2k)^{\alpha_2 j}}{(2k)^{\alpha_1 r j}}\|x\|^r \quad (3.8)$$

for all nonneg ative in tegers} p, l with $p > l$ and all $x \in \mathbf{X}$. It follows from (3.8) that the sequence $\left\{(2k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ is a cauchy sequence for all x ∈ X. Since Y is complete, the sequence $\left\{(2k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ coverges. So one can define the mapping φ: X → Y by $\phi(x) := \lim_{n\to\infty}(2k)^n f\left(\frac{x}{(2k)^n}\right)$ for all x ∈ X. Moreover, letting l = 0 and passing the limit m → ∞ in (3.8), we get (3.5). It follows from (3.4) that

$$\Big\|\psi\big(\sum_{j=1}^k x_j + \sum_{j=1}^k y_j + \sum_{j=1}^k z_j\big) + \sum_{j=1}^k \psi(x_j - y_j) - \sum_{j=1}^k \psi(z_j) - 2\sum_{j=1}^k \psi(x_j)\Big\|$$

$$= \lim_{n\to\infty} (2k)^{\alpha_2 n}\Big\| f\Big(\frac{1}{(2k)^n}\big(\sum_{j=1}^k x_j + \sum_{j=1}^k y_j + \sum_{j=1}^k z_j\big)\Big) + \sum_{j=1}^k f\big(\frac{x_j - y_j}{(2k)^n}\big) - \sum_{j=1}^k f\big(\frac{1}{(2k)^n} z_j\big)$$

$$- 2\sum_{j=1}^k f\big(\frac{1}{(2k)^n} x_j\big)\Big\|$$

$$\le \lim_{n\to\infty}\Bigg(\Big\|\lambda_1\Big(f\Big(\frac{1}{(2k)^n}\big(\sum_{j=1}^k x_j + \sum_{j=1}^k y_j + \sum_{j=1}^k z_j\big)\Big) - \sum_{j=1}^k f\big(\frac{1}{(2k)^n} x_j\big) - \sum_{j=1}^k f\big(\frac{1}{(2k)^n} y_j\big)$$

$$- \sum_{j=1}^k f\big(\frac{1}{(2k)^n} z_j\big)\Big)\Big\|$$

$$+ \Big\|\lambda_2\Big(f\Big(\frac{1}{(2k)^n}\big(\sum_{j=1}^k x_j + \sum_{j=1}^k y_j + \sum_{j=1}^k z_j\big)\Big) - \sum_{j=1}^k f\big(\frac{x_j + y_j}{(2k)^n}\big) - \sum_{j=1}^k f\big(\frac{1}{(2k)^n} z_j\big)\Big)\Big\|\Bigg)$$

$$+ \lim_{n\to\infty}\frac{|2k|^{\alpha_2 n}}{|2k|^{n\alpha_1 r}}\theta\Big(\sum_{j=1}^k \|x_j\|^r + \sum_{j=1}^k \|y_j\|^r + \sum_{j=1}^k \|z_j\|^r\Big)$$

$$= \Big\|\lambda_1\Big(\psi\big(\sum_{j=1}^k x_j + \sum_{j=1}^k y_j + \sum_{j=1}^k z_j\big) - \sum_{j=1}^k \psi(x_j) - \sum_{j=1}^k \psi(y_j) - \sum_{j=1}^k \psi(z_j)\Big)\Big\|$$

$$+ \Big\|\lambda_2\Big(\psi\big(\sum_{j=1}^k x_j + \sum_{j=1}^k y_j + \sum_{j=1}^k z_j\big) - \sum_{j=1}^k \psi(x_j + y_j) - \sum_{j=1}^k \psi(z_j)\Big)\Big\| \quad (3.9)$$

for all $x_j, y_j, z_j \in X$ for all j = 1 → n. Hence

$$\Big\|\psi\big(\sum_{j=1}^k x_j + \sum_{j=1}^k y_j + \sum_{j=1}^k z_j\big) + \sum_{j=1}^k \psi(x_j - y_j) - \sum_{j=1}^k \psi(z_j) - 2\sum_{j=1}^k \psi(x_j)\Big\|$$

$$\le \Big\|\lambda_1\Big(\psi\big(\sum_{j=1}^k x_j + \sum_{j=1}^k y_j + \sum_{j=1}^k z_j\big) - \sum_{j=1}^k \psi(x_j) - \sum_{j=1}^k \psi(y_j) - \sum_{j=1}^k \psi(z_j)\Big)\Big\|$$

$$+ \Big\|\lambda_2\Big(\psi\big(\sum_{j=1}^k x_j + \sum_{j=1}^k y_j + \sum_{j=1}^k z_j\big) - \sum_{j=1}^k \psi(x_j + y_j) - \sum_{j=1}^k \psi(z_j)\Big)\Big\| \quad (3.10)$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$. So by lemma 3.1 it follows that the mapping $\psi : \mathbf{X} \to \mathbf{Y}$ is additive. Now we need to prove unic

$$\Big\|\psi(x) - \phi'(x)\Big\| = (2k)^{\alpha_2 n}\Big\|\psi\big(\frac{x}{(2k)^n}\big) - \phi'\big(\frac{x}{(2k)^n}\big)\Big\|$$

$$\le (2k)^{\alpha_2 n}\Big(\Big\|\psi\big(\frac{x}{(2k)^n}\big) - f\big(\frac{x}{(2k)^n}\big)\Big\| + \Big\|\phi'\big(\frac{x}{(2k)^n}\big) - f\big(\frac{x}{(2k)^n}\big)\Big\|\Big)$$

$$\le \frac{4k.(2k)^{\alpha_2 n}}{(1-|\lambda_1|^{\alpha_2}-|\lambda_2|^{\alpha_2})(2k)^{\alpha_1 n r}((2k)^{\alpha_1 r} - (2k)^{\alpha_2})}\theta\|x\|^r \quad (3.11)$$

which tends to zero as n → ∞ for all x ∈ X. So, we can conclude that $\psi(x) = \varphi(x)$ for all $x \in X$. This proves thus the mapping $\psi: X \to Y$ is a unique mapping satisfying (3.5) as we expected.

Theorem 3.3. *suppose* $r < \frac{\alpha_2}{\alpha_1}$, θ *be nonngative real number, and let* $f: \mathbf{X} \to \mathbf{Y}$ *be a mapping such that*

$$\begin{aligned}&\left\|f\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\Big) + \sum_{j=1}^{k} f(x_j - y_j) - \sum_{j=1}^{k} f(z_j) - 2\sum_{j=1}^{k} f(x_j)\right\| \\ &\le \left\|\lambda_1\Big(f\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k} f(x_j) - \sum_{j=1}^{k} f(y_j) - \sum_{j=1}^{k} f(z_j)\Big)\right\| \\ &\quad + \left\|\lambda_2\Big(f\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k} f(x_j + y_j) - \sum_{j=1}^{k} f(z_j)\Big)\right\| \\ &\quad + \theta\Big(\sum_{j=1}^{k} \|x_j\|^r + \sum_{j=1}^{k} \|y_j\|^r + \sum_{j=1}^{k} \|z_j\|^r\Big) \end{aligned} \tag{3.12}$$

for all $x_j, y_j, z_j \in X$ *for all* $j = 1 \to n$. *Then there exists a unique mapping* $\psi: X \to Y$ *such that*

$$\left\|f(x) - \psi(x)\right\| \le \frac{2k}{(1 - |\lambda_1|^{\alpha_2} - |\lambda_2|^{\alpha_2})((2k)^{\alpha_2} - (2k)^{\alpha_1 r})}\theta\|x\|^r. \tag{3.13}$$

for all $x \in \mathbf{X}$

The rest of the proof is similar to the proof of Theorem 3.2.

4. ESTABLISHING THE SOLUTION FOR (1.2) IN (α_1, α_2)-HOMOGENEOUS F-SPACES

4.1. Condition for existence of solutions for Equation (1.2). Here pay attention that **X** is a α_1-homogeneous F-spaces and that **Y** is a α_2-homogeneous F-spaces.

Lemma 4.1. *Let a mapping* $f: \mathbf{X} \to \mathbf{Y}$ *satilies*

$$\begin{aligned}&\left\|f\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k} f(x_j) - \sum_{j=1}^{k} f(y_j) - \sum_{j=1}^{k} f(z_j)\right\| \\ &\le \left\|\lambda_1\Big(f\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\Big) + \sum_{j=1}^{k} f(x_j - y_j) - \sum_{j=1}^{k} f(z_j) - 2\sum_{j=1}^{k} f(x_j)\Big)\right\| \\ &+ \left\|\lambda_2\Big(f\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k} f(x_j + y_j) - \sum_{j=1}^{k} f(z_j)\Big)\right\| \end{aligned} \tag{4.1}$$

We replacing $(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k)$ *by* $(0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0)$ *in (4.1), we have*

$$\left\|(3k-1)f(0)\right\| \le \left\|\lambda_1(2k-1)f(0)\right\| + \left\|\lambda_2(2k-1)f(0)\right\|$$

Therefore

$$\Big(\left|3k-1\right|^{\beta_2} - \left|\lambda_1(2k-1)\right|^{\beta_2} - \left|\lambda_2(2k-1)\right|^{\beta_2}\Big)\left\|f(0)\right\| \le 0$$

So $f(0) = 0$.

Replacing $(x_1 \ldots, x_k, y_1, y_2, \ldots, y_k, z_1, \ldots, z_k)$ by $(0, \ldots, 0, -y, 0, \ldots, 0, 0, \ldots, 0)$, in (4.1), we get

$$\left\|f(-y) - f(-y)\right\| - \left|\lambda_1\right|\left\|f(-y) + f(y)\right\| - \left|\lambda_2\right|\left\|f(-y) - f(-y)\right\| \le 0$$

and so f is an odd mapping.

Replacing (x_1, ... , x_k, y_1, ..., y_k, z_1, ..., z_k) by (x_1,..., x_k, 0, ..., 0, z_1, ..., z_k) in (4.1) we have

$$\left\|f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}z_j)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(x_j)\right\|$$
$$\leq\left\|\lambda_1\Big(f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}z_j)-\sum_{j=1}^{k}f(x_j)-\sum_{j=1}^{k}f(z_j)\Big)\right\|$$
$$+\left\|\lambda_2\Big(f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}z_j)-\sum_{j=1}^{k}f(x_j)-\sum_{j=1}^{k}f(z_j)\Big)\right\| \quad (4.2)$$

And so

$$(1-\left|\lambda_1\right|^{\beta_2}-\left|\lambda_2\right|^{\beta_2})\left\|f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}z_j)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(x_j)\right\|\leq 0 \quad (4.3)$$

And so

$$f\Big(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}z_j\Big)=\sum_{j=1}^{k}f\Big(x_j\Big)+\sum_{j=1}^{k}f(z_j)$$

for all x_j, $z_j \in X$ for $j = 1 \to k$, as we expected.

4.2. Constructing a solution for (1.2). Now, we first study the solutions of (1.2). Note that for these inequalities, when **X** is a α_1-homogeneous F -spaces and that **Y** is a α_2-homogeneous F -spaces. Under this setting, we can show that the mapping satisfying (1.2) is additive. These results are give in the following.

Theorem 4.2. *suppose* $r>\frac{\alpha_2}{\alpha_1}$, θ *be nonngative real number, and let f* : **X** → **Y** *be a mapping such that*

$$\left\|f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j)-\sum_{j=1}^{k}f(x_j)-\sum_{j=1}^{k}f(y_j)-\sum_{j=1}^{k}f(z_j)\right\|$$
$$\leq\left\|\lambda_1\Big(f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j)+\sum_{j=1}^{k}f(x_j-y_j)-\sum_{j=1}^{k}f(z_j)-2\sum_{j=1}^{k}f(x_j)\Big)\right\|$$
$$+\left\|\lambda_2\Big(f(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j)-\sum_{j=1}^{k}f(x_j+y_j)-\sum_{j=1}^{k}f(z_j)\Big)\right\|$$
$$+\theta\Big(\sum_{j=1}^{k}\|x_j\|^r+\sum_{j=1}^{k}\|y_j\|^r+\sum_{j=1}^{k}\|z_j\|^r\Big) \quad (4.4)$$

for all x_j, y_j, $z_j \in X$ *for* all j = 1 → n. Then there exists a unique mapping ψ: X → Y such that

$$\left\|f(x)-\psi(x)\right\|\leq\frac{2k}{(1-|\lambda_1|^{\alpha_2}-|\lambda_2|^{\alpha_2})((2k)^{\alpha_1 r}-(2k)^{\alpha_2})}\theta\|x\|^r. \quad (4.5)$$

for all $x \in X$

Proof. Assume that *f*: X → Y satisfies (4.4).

Replacing (x_1,..., x_k, y_1, ..., y_k, z_1, ..., z_k) by (0,., 0,0,..., 0, 0,., 0) in (4.4),we have

So *f* (0) = 0.

Next we replacing (x1,..., xk, y1, ..., yk, z1, ..., zk) by (x,..., x, 0, ..., 0, x, ..., x) in (4.4), we get

$$\left\|f(2kx)-2kf(x)\right\| \leq \frac{2k\theta}{(1-|\lambda_1|^{\alpha_2}-|\lambda_2|^{\alpha_2})}\left\|x\right\|^r \tag{4.6}$$

for all $x \in X$. Thus

$$\left\|f(x)-2kf(\frac{x}{2k})\right\| \leq \frac{2k\theta}{|2k|^{\alpha_1 r}(1-|\lambda_1|^{\alpha_2}-|\lambda_2|^{\alpha_2})}\left\|x\right\|^r \tag{4.7}$$

for all $x \in X$. So

$$\begin{aligned}\left\|(2k)^l f(\frac{x}{(2k)^l})-(2k)^p f(\frac{x}{(2k)^p})\right\| &\leq \sum_{j=l}^{p-1}\left\|(2k)^j f(\frac{x}{(2k)^j})-(2k)^{j+1}f(\frac{x}{(2k)^{j+1}})\right\| \\ &\leq \frac{2k\theta}{|2k|^{\alpha_1 r}(1-|\lambda_1|^{\alpha_2}-|\lambda_2|^{\alpha_2})}\sum_{j=l}^{p-1}\frac{(2k)^{\alpha_2 j}}{(2k)^{\alpha_1 r j}}\left\|x\right\|^r \end{aligned} \tag{4.8}$$

for all nonnegative integers p, l with $p > l$ and all $x \in X$. It follows from (4.8) that the sequence $\{(2k)^n f(\frac{x}{(2k)^n})\}$ is a cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{(2k)^n f(\frac{x}{(2k)^n})\}$ coverges. So one can define the mapping $\varphi: X \to Y$ by $\phi(x) := \lim_{n\to\infty}(2k)^n f(\frac{x}{(2k)^n})$ for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.8), we get (4.5). It follows from (4.4) that

$$\begin{aligned}&\left\|\psi(\sum_{j=1}^k x_j+\sum_{j=1}^k y_j+\sum_{j=1}^k z_j)-\sum_{j=1}^k\psi(x_j)-\sum_{j=1}^k\psi(y_j)-\sum_{j=1}^k\psi(z_j)\right\| \\ &=\lim_{n\to\infty}(2k)^{\alpha_2 n}\left\|f\Big(\frac{1}{(2k)^n}\Big(\sum_{j=1}^k x_j+\sum_{j=1}^k y_j+\sum_{j=1}^k z_j\Big)\Big)-\sum_{j=1}^k f(\frac{x_j}{(2k)^n})-\sum_{j=1}^k f(\frac{1}{(2k)^n}y_j)\right. \\ &\left.-\sum_{j=1}^k f(\frac{1}{(2k)^n}z_j)\right\| \\ &\leq \lim_{n\to\infty}\Bigg(\left\|\lambda_1\Big(f\Big(\frac{1}{(2k)^n}\Big(\sum_{j=1}^k x_j+\sum_{j=1}^k y_j+\sum_{j=1}^k z_j\Big)\Big)+\sum_{j=1}^k f(\frac{x_j-y_j}{(2k)^n})-\sum_{j=1}^k f(\frac{1}{(2k)^n}z_j)\right. \\ &\left.-2\sum_{j=1}^k f(\frac{1}{(2k)^n}x_j)\Big)\right\| \\ &+\left\|\lambda_2\Big(f\Big(\frac{1}{(2k)^n}\Big(\sum_{j=1}^k x_j+\sum_{j=1}^k y_j+\sum_{j=1}^k z_j\Big)\Big)-\sum_{j=1}^k f(\frac{x_j+y_j}{(2k)^n})-\sum_{j=1}^k f(\frac{1}{(2k)^n}z_j)\Big)\right\|\Bigg) \\ &\quad+\lim_{n\to\infty}\frac{|2k|^{\alpha_2 n}}{|2k|^{n\alpha_1 r}}\theta\Big(\sum_{j=1}^k\|x_j\|^r+\sum_{j=1}^k\|y_j\|^r+\sum_{j=1}^k\|z_j\|^r\Big) \\ &\quad=\left\|\lambda_1\Big(\psi(\sum_{j=1}^k x_j+\sum_{j=1}^k y_j+\sum_{j=1}^k z_j)+\sum_{j=1}^k\psi(x_j-y_j)-\sum_{j=1}^k\psi(z_j)-2\sum_{j=1}^k\psi(x_j)\Big)\right\| \\ &\quad+\left\|\lambda_2\Big(\psi(\sum_{j=1}^k x_j+\sum_{j=1}^k y_j+\sum_{j=1}^k z_j)-\sum_{j=1}^k\psi(x_j+y_j)-\sum_{j=1}^k\psi(z_j)\Big)\right\| \end{aligned} \tag{4.9}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$. Hence

$$\begin{aligned}
&\left\|\psi\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k}\psi(x_j) - \sum_{j=1}^{k}\psi(y_j) - \sum_{j=1}^{k}\psi(z_j)\right\| \\
&\leq \left\|\lambda_1\Big(\psi\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\Big) + \sum_{j=1}^{k}\psi(x_j - y_j) - \sum_{j=1}^{k}\psi(z_j) - 2\sum_{j=1}^{k}\psi(x_j)\Big)\right\| \\
&+ \left\|\lambda_2\Big(\psi\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k}\psi(x_j + y_j) - \sum_{j=1}^{k}\psi(z_j)\Big)\right\| \qquad (4.10)
\end{aligned}$$

for all $x_j, y_j, z_j \in X$ *for* all $j = 1 \to n$. So by lemma 4.1 it follows that the mapping ψ: **X** $\to$ **Y** is additive. Now we need to prove uniqueness, suppose $\varphi^{J:}$ **X** $\to$ **Y** is also an additive mapping that satisfies (4.5). Then we have

$$\begin{aligned}
\left\|\psi(x) - \phi'(x)\right\| &= (2k)^{\alpha_2 n}\left\|\psi\Big(\frac{x}{(2k)^n}\Big) - \phi'\Big(\frac{x}{(2k)^n}\Big)\right\| \\
&\leq (2k)^{\alpha_2 n}\Big(\left\|\psi\Big(\frac{x}{(2k)^n}\Big) - f\Big(\frac{x}{(2k)^n}\Big)\right\| + \left\|\phi'\Big(\frac{x}{(2k)^n}\Big) - f\Big(\frac{x}{(2k)^n}\Big)\right\|\Big) \\
&\leq \frac{4k.(2k)^{\alpha_2 n}}{(1 - |\lambda_1|^{\alpha_2} - |\lambda_2|^{\alpha_2})(2k)^{\alpha_1 n r}((2k)^{\alpha_1 r} - (2k)^{\alpha_2})}\theta\|x\|^r \qquad (4.11)
\end{aligned}$$

Theorem 4.3. *suppose* $r < \frac{\alpha_2}{\alpha_1}$, θ *be nonngative real number, and let* f : **X** $\to$ **Y** *be a mapping such that*

$$\begin{aligned}
&\left\|f\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k}f(x_j) - \sum_{j=1}^{k}f(y_j) - \sum_{j=1}^{k}f(z_j)\right\| \\
&\leq \left\|\lambda_1\Big(f\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\Big) + \sum_{j=1}^{k}f(x_j - y_j) - \sum_{j=1}^{k}f(z_j) - 2\sum_{j=1}^{k}f(x_j)\Big)\right\| \\
&+ \left\|\lambda_2\Big(f\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\Big) - \sum_{j=1}^{k}f(x_j + y_j) - \sum_{j=1}^{k}f(z_j)\Big)\right\| \\
&+ \theta\Big(\sum_{j=1}^{k}\|x_j\|^r + \sum_{j=1}^{k}\|y_j\|^r + \sum_{j=1}^{k}\|z_j\|^r\Big) \qquad (4.12)
\end{aligned}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$. Then there exists a unique mapping ψ: X $\to$ Ysuch that

$$\left\|f(x) - \psi(x)\right\| \leq \frac{2k}{(1 - |\lambda_1|^{\alpha_2} - |\lambda_2|^{\alpha_2})((2k)^{\alpha_2} - (2k)^{\alpha_1 r})}\theta\|x\|^r. \qquad (4.13)$$

for all $x \in X$

The rest of the proof is similar to the proof of Theorem 4.2.

5. Conclusion

The result in this paper is that I have built the Jensen's additive (λ_1, λ_2)-function inequality with 3k-variables over (α_1, α_2)-homogeneous F spaces and I show the existence of n solutions for them.

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