STABILITY OF FUNCTIONAL INEQUALITIES WITH 3*K*-VARIABLE BASED ON JORDAN-VON NEUMANN TYPE ADDITIVE FUNCTIONAL EQUATIONS IN BANACH SPACE

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Abstract
In this paper, we study to solve the Cauchy, Jensen and Cauchy-Jensen additive function inequalities with 3k-variables related to Jordan-von Neumann type in Banach space. These are the main results of this paper.

Keywords
Normed Spaces; Banach Space; Generalized Hyers-Ulam-Rassias Stabil- ity Jordan-von Neumann-Type Additive Functional Equation; Cauchy, Jensen Additive Function Inequalities.

1. INTRODUCTION

Let X and Y be a normed spaces on the same field K, and f : X → Y be a mapping. We use the notation · X · Y for corresponding the norms on X and Y. In this paper, we investisgate additive functional inequalities associated with Jordan-Von Neumann type additive functional equatonal when X is a normed space with norm · X and that Y is a Banach space with norm · Y . In fact, when X is a normed space with norm · X and that Y is a Banach space with norm · Y we solve and prove the Hyers − Ulam − Rassias type stability of following additive functional inequalities

$$\left\|\sum_{j=1}^{k} f(x_j)+\sum_{j=1}^{k} f(y_j)+\sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}} \leq \left\|2kf\left(\frac{\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j}{2k}\right)\right\|_{\mathbf{Y}}, \tag{1.1}$$

and

$$\left\|\sum_{j=1}^{k} f(x_j)+\sum_{j=1}^{k} f(y_j)+\sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}} \leq \left\|f\left(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j\right)\right\|_{\mathbf{Y}}, \tag{1.2}$$

final

$$\left\|\sum_{j=1}^{k} f(x_j)+\sum_{j=1}^{k} f(y_j)+2k\sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}} \leq \left\|2kf\left(\frac{\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j}{2k}+\sum_{j=1}^{k} z_j\right)\right\|_{\mathbf{Y}}. \tag{1.3}$$

The study the stability of generalized additive functional inequalities associated with jordan-von neumann type additive functional equational originated from a question of S.M. Ulam[1], concerning the stability of group homomorphisms.

Let (G, ∗) be a group and let (G0 , ◦, d) be a metric group with metric (d ·, ·) . Geven ∈ > 0, does there exist a δ > 0 such that if f : G → G0 satisfies

$$d\big(f(x*y), f(x)\circ f(y)\big) < \delta, \forall x \in \mathbf{G}$$

then there is a homomorphism h : G → G

$$d\big(f(x), h(x)\big) < \epsilon, \forall x \in \mathbf{G}$$

The concept of stability for a functional equation arises when we replace functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given function equation? Hyers gave a first affirmative answes the question of Ulam as follows: In 1941 D. H. Hyers [2] Let ∈ ≥ 0 and let f : E1 → E2 be a mapping between Banach space

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon, \tag{1.4}$$

for all x, y ∈ E1 and some ∈ ≥ 0. It was shown that the limit

$$T(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n} \tag{1.5}$$

exists for all x ∈ E1 and that T : E1 → E2 is that unique additive mapping satisfying

$$\|f(x)-T(x)\| \leq \epsilon, \forall x \in \mathbf{E_1}. \tag{1.6}$$

Next in 1978 Th. M. Rassias [3] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded:
Consider E, E0 to be two Banach spaces, and let f : E → E0 be a mapping such that f tx is continous in t for each fixed x. Assume that there exist θ ≥ 0 and p ∈ [0, 1), ∈ > 0 Such that

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon(\|x\|^p+\|y\|^p), \forall x,y \in \mathbb{E}. \tag{1.7}$$

Where ∈ and p is constants with ∈ > 0 and < 1. Then the limit

$$T(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n} \tag{1.8}$$

there exists a unique linear $L : \mathbf{E} \to \mathbf{E}'$ satisfies

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p, x \in \mathbf{E}. \qquad (1.9)$$

If p < 0, then inequality (1.7) holds for x, y /= 0 and (1.9) for x /= 0

We notice that in Rassias' functional inequality (1.7) Mathematicians around the world such as [4],[5] as well as Rassias have asserted that the inequality (1.7) no longer holds true when $p = 1$ from the assertion that gave rise to the idea to generalize the generalized functional equation Hyers- Ulam more specifically.

Thus, to replace the non-existent condition mentioned above, Mathematician Rassias

[2] has given the following specific conditions: $\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^p$ for $p, q \in \mathbb{R}$ with P+Q /= 1.

for all $x. \in \mathbf{E}$ Ga˘vruta[6] provided a further generalization of Rassias theorem. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings.

Afterward Gil$a$´ny [7] showed that is if satisfies the functional inequality

$$\left\|2f(x) + 2f(y) - f(xy^{-1})\right\| \le \|f(xy)\| \qquad (1.10)$$

f satisfies the Jordan-von Newman functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}) \qquad (1.11)$$

Then, mathematicians in the world proved to extend the functional inequality (1.11) as [7]-[17].In addition, mathematicians have developed the achievements of their predecessors who have built mathematical models from advanced to modern mathematics, especially functional equations applied on function spaces to Unlocking means connecting with other Maths. [3]-[35]Recently, the authors studied the Hyers-Ulam-Rassias type stability for the following functional inequalities (see [31],[32],[34])

$$\left\|f(x) + f(y) + f(z)\right\| \le \left\|k\left(f\left(\frac{x+y+z}{k}\right)\right)\right\|, |k| < |3|, \qquad (1.12)$$

$$\left\|f(x_1) + f(x_2) + \cdots + f(x_n)\right\| \le \left\|kf\left(\frac{x_1 + x_2 + \cdots x_n}{k}\right)\right\|, |n| > |k|, \qquad (1.13)$$

$$\left\|\sum_{j=1}^{n} f(x_j) + \sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\| \le \left\|kf\left(\frac{\sum_{j=1}^{n} x_j}{k} + \frac{\sum_{j=1}^{n} x_{n+j}}{n \cdot k}\right)\right\|, |n| > |k|. \qquad (1.14)$$

In banach spaces.

In this paper, we solve and proved the Hyers-Ulam- Rassias type stability for functional inequalitie (1.1). (1.2) and (1.3) ie the functional inequalities with 3k-variables. Under suitable assumptions on spaces **X** and **Y,** we will prove that the mappings satisfying the functional inequatilie (1.1). (1.2) and (1.3). Thus, the results in this paper are general- ization of those in [21],[31],[32],[34] for functional inequatilie with 3k-variables.
The paper is organized as followns:
In section preliminarier we remind some basic notations in such as Solutions of the in- equalities.

Section:3 The basis for building solutions for functional inequalities related to the type of Jordan-Neuman additive functional equations
Section:4 Establishing solutions to functional inequality (1.1) related to the type of Jensen additive functional equation
Section:5 Establishing solutions to functional inequality (1.2) related to the type of Cauchy additive functional equation.
Section:6 Establishing solutions to functional inequality (1.3) related to the type of Cauchy-Jensen additive functional equation.

2. PRELIMINARIES

2.1. **Solutions of the inequalities.** The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the Cauchuy equation. In particular, every solution of the cauchuy equation is said to be an *additive mapping*.

The functional equation

$$f(\frac{x+y}{2})=\frac{1}{2}f(x)+\frac{1}{2}f(y)$$

is called the Jensen equation. In particular, every solution of the Jensen equation is said to be a Jensen *additive mapping*.
The functional equation

$$2f(\frac{x+y}{2}+z)=f(x)+f(y)+2f(z)$$

is called the Cauchuy-Jensen equation. In particular, every solution of the Cauchuy- Jensen equation is said to be a Jensen-Cauchy *additive mapping*.

3. THE BASIS FOR BUILDING SOLUTIONS FOR FUNCTIONAL INEqUALITIES RELATED TO THE TYPE OF JORDAN-NEUMAN ADDITIVE FUNCTIONAL EqUATIONS

The basis for building solutions for functional inequalities related to the type of Jordan- Neuman additive functional equations. Now, we first study the solutions of (1.1), (1.2) and (1.3). Note that for this inequalities, X is a normed space with norm · **X and** that **Y** is a *Banach* space with norm · **Y**. Under this setting, we can show that the mappings satisfying (1.1), (1.2) and (1.3) is additive.
Here we assume that G is a 3k-divisible abelian group.

Proposition 3.1. *Suppose f:* **X** → **Y** *be a mapping such that*

$$\left\|\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+\sum_{j=1}^{k}f(z_j)\right\|_{\mathbf{Y}}\le\left\|2kf\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j}{2k}\right)\right\|_{\mathbf{Y}}\quad(3.1)$$

for all x_j, y_n,z_j ∈ **X** *for all* $j = 1 \to n$ *then f is additive.*

Proof. Assume that f : **X** → **Y** satisfies (3.1).
We replacing $(x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots,z_k)$ by $0,\dots,0,0,\dots,0,0,\dots,0)$ in (3.1), we have
$f(0)=0$.
Next We replacing $(x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots,z_k)$ by $x,0,\dots,0,-x,0\dots,0,0,\dots,0)$ in (3.1), we have

$$\left\|f(x)+f(-x)\right\|_{\mathbf{Y}}\le\left\|2nf(0)\right\|_{\mathbf{Y}}\quad(3.2)$$

, for all $x \in$ **X**.
Hence. $f(x)=-f(-x),\ \forall x\in\mathbf{X}$
Next We replacing $(x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots,z_k)$ by $x,0,\dots,0,y,0\dots,0,-x-y,\dots,0)$ in (3.1), we have

$$\left\|f(x)+f(y)-f(x+y)\right\|_{Y}=\left\|f(x)+f(y)+f(-x-y)\right\|_{Y}\le\left\|2nf(0)\right\|_{Y}=0\quad(3.3)$$

for all $x,y \in X$. It follows that $f(x+y)=f(x)+f(y)$ This completes the proof.

Proposition 3.2. *f:* **X** → **Y** *be a mapping such a that*

$$\left\|\sum_{j=1}^{n}f(x_j)+\sum_{j=1}^{n}f(y_j)+\sum_{j=1}^{n}f(z_j)\right\|_{\mathbf{Y}}\le\left\|f\Big(\sum_{j=1}^{n}x_j+\sum_{j=1}^{n}y_j+\sum_{j=1}^{n}z_j\Big)\right\|_{\mathbf{Y}}\quad(3.4)$$

for all x_j, y_j,z_j ∈ **X** *for all* $j = 1 \to n$ *then f is additive.*
Proof. Assume that f : $X \to$Y satisfies (3.4).

We replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by $0, ..., 0, 0, ..., 0, 0, ..., 0)$ in (3.4), we have $f(0) = 0$.

Next We replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by $(x, ..., 0, -x, .., 0, 0, ..., 0)$ in (3.4), we have

$$\left\|f(x) + f(-x)\right\|_{\mathbf{Y}} \leq \left\|f(0)\right\|_{\mathbf{Y}} \tag{3.5}$$

for all $x \in \mathbf{X}$.

Hence. $f(x) = -f(-x), \forall x \in \mathbf{X}$

Next We replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by $x, 0, ..., 0, y, 0..., 0, -x - y, ..., 0)$ in (3.4), we have

$$\left\|f(x) + f(y) - f(x+y)\right\|_{\mathbf{Y}} = \left\|f(x) + f(y) + f(-x-y)\right\|_{Y} \leq \left\|f(0)\right\|_{\mathbf{Y}} = 0 \tag{3.6}$$

for all $x,y \in \mathbf{X}$. It follows that $f(x+y) = f(x) + f(y)$ This completes the proof.

Proposition 3.3. *f: $\mathbf{G} \to \mathbf{Y}$ be a mapping such that*

$$\left\|\sum_{j=1}^{n} f(x_j) + \sum_{j=1}^{n} f(y_j) + 2n\sum_{j=1}^{n} f(z_j)\right\|_{\mathbf{Y}} \leq \left\|2nf\left(\frac{\sum_{j=1}^{n} x_j + \sum_{j=1}^{n} y_j}{2n} + \sum_{j=1}^{n} z_j\right)\right\|_{\mathbf{Y}} \tag{3.7}$$

for all x_j, y_j,$z_j \in \mathbf{X}$ for all $j = 1 \to n$ then f is additive.

Proof. Assume that f : $\mathbf{G} \to \mathbf{Y}$ satisfies (3.7).

We replacing $(x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_n)$ by $0, ..., 0, 0, ..., 0, 0, ..., 0)$ in (3.7), we have we get

$$\left\|(2n^2 + 2n)f(0)\right\| \leq \left\|2nf(0)\right\|_{\mathbf{Y}} \tag{3.8}$$

. So $f(0) = 0$

Next We replacing $(x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_n)$ by $(x, 0, ..., 0, -x, 0..., 0, 0, ..., 0)$ in (3.7), we have

$$\left\|f(x) + f(-x)\right\|_{Y} \leq \left\|2nf(0)\right\|_{\mathbf{Y}} \tag{3.9}$$

for all $x \in$X.

Hence. $f(x) = -f(-x), \forall x \in \mathbf{X}$

Next We replacing $(x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_n)$ by $-2nz, 0, ..., 0, 0..., 0, z, 0, ..., 0)$ in (3.7), we have

$$\left\|f(-2nz) + 2nf(z)\right\|_{\mathbf{Y}} \leq \left\|2nf(0)\right\|_{\mathbf{Y}} \tag{3.10}$$

for all $x \in \mathbf{X}$.

Thus $f(2nz) = 2nf(z), \forall z \in \mathbf{G}$.

Next We replacing $(x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_n)$ by $(x_1, ..., x_n, y_1, ..., y_n, -\frac{x_1+y_1}{2n}, ..., -\frac{x_n+y_n}{2n})$ in (3.7), we have

$$\begin{aligned}&\left\|\sum_{j=1}^{n} f(x_j) + \sum_{j=1}^{n} f(y_j) - \sum_{j=1}^{n} f(x_j + y_j)\right\|_{\mathbf{Y}} \\ &\qquad \left\|\sum_{j=1}^{n} f(x_j) + \sum_{j=1}^{n} f(y_j) + 2n\sum_{j=1}^{n} f(-\frac{x_j + y_j}{2n})\right\|_{\mathbf{Y}} \\ &\qquad \leq \left\|2nf(0)\right\|_{\mathbf{Y}}\end{aligned} \tag{3.11}$$

$\forall x_1, ..., x_k, y_1, ..., y_k, -\frac{x_1+y_1}{2n}, ..., -\frac{x_k+y_k}{2n} \in \mathbf{G}$ Thus

$$\sum_{j=1}^{n} f(x_j) + \sum_{j=1}^{n} f(y_j) - \sum_{j=1}^{n} f(x_j + y_j) = 0 \tag{3.12}$$

Next put $x = x_j, y = y_j$ for all $j = 1 \to n$ in (3.12), we have

$$f(x+y) = f(x) + f(y)$$

for all $x,y \in \mathbf{G}$. It follows that f is an additive maping and the proof is complete.

4. Establishing solutions to functional inequality (1.1) related to the type of Jensen additive functional equation

Now, we first study the solutions of (1.1). Note that for this inequality, **X** is a normed space with normand that **Y** is a *Banach* space with norm. Under this setting, we can show that the mappings satisfying (1.1) is Jensen additive. These results are give in the following.

Theorem 4.1. *Suppose $q > 1$, θ be non-negative real and $f : \mathbf{X} \to \mathbf{Y}$ be an odd mapping such that*

$$\begin{aligned}\left\|\sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f(y_j) + \sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}} &\le \left\|2kf\left(\frac{\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j}{2k}\right)\right\|_{\mathbf{Y}} \\ &+ \theta\left(\sum_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{q} + \sum_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^{q} + \sum_{j=1}^{k}\left\|z_j\right\|_{\mathbf{X}}^{q}\right)\end{aligned} \tag{4.1}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for all $j = 1 \to n$

. Then there exists a unique additive mapping $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|f(x) - H(x)\right\|_{Y} \le \frac{(2k)^q + 2k}{(2k)^q - 2k}\theta\left\|x\right\|_{X}^{q}. \tag{4.2}$$

for all $x \in \mathbf{X}$.

Proof. Assume that $f : \mathbf{X} \to \mathbf{Y}$ satisfies (4.1).

We replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by $(x, ..., x, x, ..., x, -2kx, 0, ..., 0)$ in (4.1), we Have

$$\left\|2kf(x) - f(2kx)\right\|_{\mathbf{Y}} = \left\|2kf(x) + f(-2kx)\right\|_{\mathbf{Y}} \le \left((2k)^q + 2k\right)\theta\left\|x\right\|_{\mathbf{X}}^{q} \tag{4.3}$$

for all $x \in \mathbf{X}$.So

$$\left\|f(x) - 2kf\left(\frac{x}{2k}\right)\right\| \le \frac{(2k)^q + 2k}{(2k)^q}\frac{\theta}{k}\left\|x\right\|_{\mathbf{X}}^{q} \tag{4.4}$$

Hence we have

$$\begin{aligned}\left\|(2k)^l f\left(\frac{x}{(2k)^l}\right) - (2k)^m f\left(\frac{x}{(2k)^m}\right)\right\|_{Y} &\le \sum_{j=l}^{m-1}\left\|(2k)^j f\left(\frac{x}{(2k)^j}\right) - (2k)^{j+1} f\left(\frac{x}{(2k)^{j+1}}\right)\right\|_{Y} \\ &\le \frac{(2k)^q + 2k}{(2k)^q}\frac{\theta}{k}\sum_{j=l}^{m-1}\frac{(2k)^j}{(2k)^{qj}}\theta\left\|x\right\|_{\mathbf{X}}^{q}.\end{aligned} \tag{4.5}$$

for all nonnegative m and l with $m > l$, $\forall x \in \mathbf{X}$. It follows from (4.5) that the sequence $\left\{(2k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ is a cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is a Banach space, the sequence $\left\{(2k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ coverges.

So one can define the mapping $H : \mathbf{X} \to \mathbf{Y}$ by

$$H(x) := \lim_{n\to\infty} (2k)^n f\left(\frac{x}{(2k)^n}\right)$$

for all $x \in \mathbf{X}$.

$$\begin{aligned}
&\left\|\sum_{j=1}^{k} H(x_j) + \sum_{j=1}^{k} H(y_j) + \sum_{j=1}^{k} H(z_j)\right\|_{\mathbf{Y}} \\
&= \lim_{n\to\infty} (2k)^n \left\|\sum_{j=1}^{k} f\left(\frac{x_j}{(2k)^n}\right) + \sum_{j=1}^{k} f\left(\frac{y_j}{(2k)^n}\right) + \sum_{j=1}^{k} f\left(\frac{z_j}{(2k)^n}\right)\right\|_{\mathbf{Y}} \\
&\le \lim_{n\to\infty} (2k)^n \left\|2kf\left(\frac{\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j}{(2k)^{n+1}}\right)\right\|_{\mathbf{Y}} \\
&+ \lim_{n\to\infty} \frac{(2k)^n}{(2k)^{kq}} \theta \left(\sum_{j=1}^{k} \|x_j\|_{\mathbf{X}}^q + \sum_{j=1}^{k} \|y_j\|_{\mathbf{X}}^q + \sum_{j=1}^{k} \|z_j\|_{\mathbf{X}}^q\right) \\
&= \left\|2kH\left(\frac{\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j}{2k}\right)\right\|_{\mathbf{Y}} \qquad (4.6)
\end{aligned}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for all $j = 1 \to k$. So

$$\begin{aligned}
&\left\|\sum_{j=1}^{k} H(x_j) + \sum_{j=1}^{k} H(y_j) + \sum_{j=1}^{k} H(z_j)\right\|_{\mathbf{Y}} \\
&\qquad \le \left\|2kH\left(\frac{\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j}{2k}\right)\right\|_{\mathbf{Y}} \qquad (4.7)
\end{aligned}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for all $j = 1 \to n$. By Proposition 3.1, the mapping $H : \mathbf{X} \to \mathbf{Y}$ is additive. Now, let $T : \mathbf{X} \to \mathbf{Y}$ be another additive mapping satistify (4.2) then we have

$$\begin{aligned}
\|H(x) - T(x)\|_{\mathbf{Y}} &= (2k)^n \left\|h\left(\frac{x}{(2k)^n}\right) - T\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}} \\
&\le (2k)^n \left(\left\|h\left(\frac{x}{(2k)^n}\right) - f\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}} + \left\|T\left(\frac{x}{(2k)^n}\right) - f\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}}\right) \\
&\le \frac{2((2k)^n + 2k)}{(2k)^n - 2k} \cdot \frac{\theta}{k} \cdot \frac{(2k)^n}{(2k)^{nq}} \|x\|_{\mathbf{X}}^q. \qquad (4.8)
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in \mathbf{X}$. So we can conclude that $H(x) = T(x)$ for all $x \in \mathbf{X}$. This proves the uniqueness of H. Thus the mapping $H : \mathbf{X} \to \mathbf{Y}$ is additive mapping satisfying (4.2).

Theorem 4.2. *Suppose $q < 1$, θ be positive real numbers and $f : \mathbf{X} \to \mathbf{Y}$ be an odd mapping such that*

$$\left\|\sum_{j=1}^{k} f(x_j)+\sum_{j=1}^{k} f(y_j)+\sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}} \le \left\|2kf\left(\frac{\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j}{2k}\right)\right\|_{\mathbf{Y}} + \theta\left(\sum_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{q}+\sum_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^{q}+\sum_{j=1}^{k}\left\|z_j\right\|_{\mathbf{X}}^{q}\right) \tag{4.9}$$

for all x_j, $y_j, z_j \in \mathbf{X}$ for all $j = 1 \to n$
. Then there exists a unique additive mapping $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|f(x)-H(x)\right\|_{Y} \le \frac{2k+(2k)^{q}}{2k-(2k)^{q}}\theta\left\|x\right\|_{X}^{q}. \tag{4.10}$$

for all $x \in \mathbf{X}$.
Notice that: Form (4.3) we have

$$\left\|f(x)-\frac{1}{2k}f(2kx)\right\|_{\mathbf{Y}} \le \frac{2k+(2k)^{q}}{2k}\theta\left\|x\right\|_{\mathbf{X}}^{q} \tag{4.11}$$

for all $x \in \mathbf{X}$. The rest of the proof is similar to the proof of Theorem 4.1.

Theorem 4.3. *Suppose $q > p^{-1}$ with $p \ge 3$, θ be non-negative real and $f : \mathbf{X} \to \mathbf{Y}$ be an odd mapping such that*

$$\left\|\sum_{j=1}^{k} f(x_j)+\sum_{j=1}^{k} f(y_j)+\sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}} \le \left\|2kf\left(\frac{\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j}{2k}\right)\right\|_{\mathbf{Y}} + \theta\prod_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{q}\cdot\prod_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^{q}\cdot\left\|z_1\right\|_{\mathbf{X}}^{kq}\cdot(1+\prod_{j=2}^{k}\left\|z_j\right\|_{\mathbf{X}}^{q}) \tag{4.12}$$

for all x_j, $y_j, z_j \in \mathbf{X}$ for all $j = 1 \to n$
. Then there exists a unique additive mapping $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|f(x)-H(x)\right\|_{Y} \le \frac{(2k)^{q}}{(2k)^{3kq}-2k}\theta\left\|x\right\|_{X}^{3kq}. \tag{4.13}$$

for all $x \in \mathbf{X}$.
Proof. Assume that $f : \mathbf{X} \to \mathbf{Y}$ satisfies (4.12).
We replacing $(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k)$ by $(x, \ldots, x, x, \ldots, x, -2kx, 0, \ldots, 0)$ in (4.12), we have

$$\left\|2kf(x)-f(2kx)\right\|_{\mathbf{Y}} = \left\|2kf(x)+f(-2kx)\right\|_{\mathbf{Y}} \le |2k|^{kq}\theta\left\|x\right\|_{\mathbf{X}}^{3kq} \tag{4.14}$$

for all $x \in \mathbf{X}$. So

$$\left\|f(x)-2kf\left(\frac{x}{2k}\right)\right\| \le \frac{|2k|^{kq}}{|2k|^{3kq}}\left\|x\right\|_{\mathbf{X}}^{3kq} \tag{4.15}$$

Hence we have

$$\begin{aligned}\Big\|(2k)^l f\Big(\frac{x}{(2k)^l}\Big)-(2k)^m f\Big(\frac{x}{(2k)^m}\Big)\Big\|_{\mathbf{Y}} &\le \sum_{j=l}^{m-1}\Big\|(2k)^j f\Big(\frac{x}{(2k)^j}\Big)-(2k)^{j+1} f\Big(\frac{x}{(2k)^{j+1}}\Big)\Big\|_{Y}\\ &\le \frac{|2k|^{kq}}{|2k|^{3kq}}\sum_{j=l}^{m-1}\frac{(2k)^j}{(2k)^{3kqj}}\theta\|x\|_{\mathbf{X}}^{3kq}. \qquad (4.16)\end{aligned}$$

for all nongnegative m and l with $m > l$, $\forall x \in \mathbf{X}$. It follows from (4.16) that the sequence $\left\{(2k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ is a cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is a Banach space, the sequence $\left\{(2k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ coverges.

So one can define the mapping $H : \mathbf{X} \to \mathbf{Y}$ by

$$H(x) := \lim_{n\to\infty}(2k)^n f\Big(\frac{x}{(2k)^n}\Big)$$

for all $x \in \mathbf{X}$. Moreover, letting $l = 0$ and passing the *limit* $m \to \infty$ in (4.16), we have (4.13). The rest of the Prooft is similar to the Proof of the Theorem 4.1.

Theorem 4.4. *Suppose $q < p^{-1}$ with $p \ge 3$, θ be non-negative real and $f : \mathbf{X} \to \mathbf{Y}$ be an odd mapping such that*

$$\begin{aligned}\Big\|\sum_{j=1}^{k} f(x_j)+\sum_{j=1}^{k} f(y_j)+\sum_{j=1}^{k} f(z_j)\Big\|_{\mathbf{Y}} &\le \Big\|2kf\Big(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j}{2k}\Big)\Big\|_{\mathbf{Y}}\\ &+\theta\prod_{j=1}^{k}\|x_j\|_{\mathbf{X}}^{q}\cdot\prod_{j=1}^{k}\|y_j\|_{\mathbf{X}}^{q}\cdot\|z_1\|_{\mathbf{X}}^{kq}\cdot\Big(1+\prod_{j=2}^{k}\|z_j\|_{\mathbf{X}}^{q}\Big) \qquad (4.17)\end{aligned}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for all $j = 1 \to n$
. Then there exists a unique additive mapping $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\|f(x)-H(x)\|_{Y} \le \frac{(2k)^q}{2k-(2k)^{3kq}}\theta\|x\|_{X}^{3kq}. \qquad (4.18)$$

for all x ∈ X.

The rest of the Prooft is similar to the Proof of the Theorem 4.1.

5.Establishing solutions to functional inequality (1.2) related to the type of Cauchy additive functional equation

Now, we first study the solutions of (1.2). Note that for this inequality, X is a normed space with norm · X and that Y is a Banach space with norm · Y . Under this setting, we can show that the mappings satisfying (1.2) is Cauchy additive. These results are give in the following.

Theorem 5.1. Suppose q > 1, θ be non-negative real and f : X → Y be an odd mapping such that

$$\begin{aligned}\Big\|\sum_{j=1}^{k} f(x_j)+\sum_{j=1}^{k} f(y_j)+\sum_{j=1}^{k} f(z_j)\Big\|_{\mathbf{Y}} &\le \Big\|f\Big(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j\Big)\Big\|_{\mathbf{Y}}\\ &+\theta\Big(\sum_{j=1}^{k}\|x_j\|_{\mathbf{X}}^{q}+\sum_{j=1}^{k}\|y_j\|_{\mathbf{X}}^{q}+\sum_{j=1}^{k}\|z_j\|_{\mathbf{X}}^{q}\Big) \qquad (5.1)\end{aligned}$$

for all x_j, y_j, $z_j \in \mathbf{X}$ *for all* $j = 1 \to n$

. *Then there exists a unique additive mapping* $H : \mathbf{X} \to \mathbf{Y}$ *such that*

$$\left\|f(x) - H(x)\right\|_{\mathbf{Y}} \leq \frac{(2k)^q + 2k}{(2k)^q - 2k}\theta\left\|x\right\|_{\mathbf{X}}^q. \quad (5.2)$$

for all $x \in \mathbf{X}$.

The rest of the Prooft is similar to the Proof of the Theorem 4.1.

Theorem 5.2. *Suppose* $q < 1$, θ *be positive real numbers and* $f : \mathbf{X} \to \mathbf{Y}$ *be a mapping such that*

$$\left\|\sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f(y_j) + \sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}} \leq \left\|f\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\right)\right\|_{\mathbf{Y}} + \theta\left(\sum_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^q + \sum_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^q + \sum_{j=1}^{k}\left\|z_j\right\|_{\mathbf{X}}^q\right) \quad (5.3)$$

for all x_j, y_j, $z_j \in \mathbf{X}$ *for all* $j = 1 \to n$

. *Then there exists a unique additive mapping* $H : \mathbf{X} \to \mathbf{Y}$ *such that*

$$\left\|\frac{f(x) + f(-x)}{2k} - H(x)\right\|_{Y} \leq \frac{2k + (2k)^q}{2k - (2k)^q}\theta\left\|x\right\|_{X}^q. \quad (5.4)$$

for all $x \in \mathbf{X}$.

The rest of the proof is similar to the proof of Theorem 4.1 and 4.2.

Theorem 5.3. *Suppose* $q > p^{-1}$ *with* $p \geq 3$, θ *be non-negative real and* $f : \mathbf{X} \to \mathbf{Y}$ *be an odd mapping such that*

$$\left\|\sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f(y_j) + \sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}} \leq \left\|f\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\right)\right\|_{\mathbf{Y}} + \theta\prod_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^q \cdot \prod_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^q \cdot \left\|z_1\right\|_{\mathbf{X}}^{kq} \cdot (1 + \prod_{j=2}^{k}\left\|z_j\right\|_{\mathbf{X}}^q) \quad (5.5)$$

for all x_j, y_j, $z_j \in \mathbf{X}$ *for all* $j = 1 \to n$

. *Then there exists a unique additive mapping* $H : \mathbf{X} \to \mathbf{Y}$ *such that*

$$\left\|f(x) - H(x)\right\|_{Y} \leq \frac{(2k)^q}{(2k)^{3kq} - 2k}\theta\left\|x\right\|_{X}^{3kq}. \quad (5.6)$$

for all $x \in \mathbf{X}$.

Proof. Assume that $f : \mathbf{X} \to \mathbf{Y}$ satisfies (5.5).

We replacing $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ by $(x, ..., x, x, ..., x, -2kx, 0, ..., 0)$ in (5.5), we have

$$\left\|2kf(x) - f(2kx)\right\|_{\mathbf{Y}} = \left\|2kf(x) + f(-2kx)\right\|_{\mathbf{Y}} \leq |2k|^{kq}\theta\left\|x\right\|_{\mathbf{X}}^{3kq} \quad (5.7)$$

for all $x \in \mathbf{X}$. The rest of the Prooft is similar to the Proof of the Theorem 4.1 and 4.3.

Theorem 5.4. *Suppose* $q < p^{-1}$ *with* $p \geq 3$, θ *be non-negative real and* $f : \mathbf{X} \to \mathbf{Y}$ *be an odd mapping such that*

$$\left\|\sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f(y_j) + \sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}} \leq \left\|2kf\left(\frac{\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j}{2k}\right)\right\|_{\mathbf{Y}} + \theta\prod_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^q \cdot \prod_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^q \cdot \left\|z_1\right\|_{\mathbf{X}}^{kq} \cdot (1 + \prod_{j=2}^{k}\left\|z_j\right\|_{\mathbf{X}}^q) \quad (5.8)$$

*for all x_j, y_j,z_j ∈ **X** for all $j = 1 \to n$*
*. Then there exists a unique additive mapping H : **X** → **Y** such that*

$$\left\|f(x)-H(x)\right\|_{\mathbf{Y}}\leq\frac{(2k)^{q}}{2k-(2k)^{3kq}}\theta\left\|x\right\|_{\mathbf{X}}^{3kq}. \tag{5.9}$$

for all x ∈ X.
The rest of the Prooft is similar to the Proof of the Theorem 4.1 and 4.4.

6. Establishing solutions to functional inequality (1.3) related to the type of Cauchy-Jensen additive functional equation
Now, we first study the solutions of (1.3). Note that for this inequality, X is a normed space with norm · X and that Y is a Banach space with norm · Y . Under this setting, we can show that the mappings satisfying (1.3) is Cauchy-Jensen additive. These results are give in the following.

Theorem 6.1. Suppose q > 1, θ be non-negative real, f(0) = 0 and f : X → Y be a mapping such that

$$\left\|\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+2k\sum_{j=1}^{k}f(z_j)\right\|_{\mathbf{Y}}\leq\left\|2kf\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j}{2k}+\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{q}+\sum_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^{q}+\sum_{j=1}^{k}\left\|z_j\right\|_{\mathbf{X}}^{q}\right) \tag{6.1}$$

for all xj , yj , zj ∈ X for all j = 1 → n. Then there exists a unique additive mapping H : X → Y such that

$$\left\|f(x)-H(x)\right\|_{Y}\leq\frac{(2k)^{q}+1}{(2k)^{q}-2k}\theta\left\|x\right\|_{X}^{q}. \tag{6.2}$$

*for all x ∈ **X**.*
Proof. Assume that f : **X** → **Y** satisfies (6.1).
We replacing $(x_1,...,x_k,y_1,...,y_k,z_1,...,z_k)$ by $2kx,0...,0,0,...,0,-x,0,...,0)$ in (6.1), we have

$$\left\|f(2kx)-2kf(x)\right\|_{\mathbf{Y}}=\left\|f(2kx)+2kf(-x)\right\|_{\mathbf{Y}}\leq\left((2k)^{q}+1\right)\theta\left\|x\right\|_{\mathbf{X}}^{q} \tag{6.3}$$

for all x ∈ **X**. So

$$\left\|f(x)-2kf\left(\frac{x}{2k}\right)\right\|\leq\frac{(2k)^{q}+1}{(2k)^{q}}\left\|x\right\|_{\mathbf{X}}^{q} \tag{6.4}$$

The rest of the proof is similar to the proof of Theorem 4.1.

Theorem 6.2. *Suppose q < 1, θ be positive real numbers and f:* **X** → **Y** *be a mapping such that*

$$\left\|\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+\sum_{j=1}^{k}f(z_j)\right\|_{\mathbf{Y}}\leq\left\|2kf\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j}{2k}+\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{q}+\sum_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^{q}+\sum_{j=1}^{k}\left\|z_j\right\|_{\mathbf{X}}^{q}\right) \tag{6.5}$$

*for all x_j, y_j,z_j ∈ **X** for all $j = 1 \to n$*

$$\left\|f(x)-H(x)\right\|_{Y}\leq\frac{2k+(2k)^{q}}{2k-(2k)^{q}}\theta\left\|x\right\|_{X}^{q}. \tag{6.6}$$

*for all x ∈ **X**.*
The rest of the proof is similar to the proof of Theorem 4.1 and 4.2.

5. Conclusion

In this paper I have given three general functional inequalities and I have shown that their solutions are determined on normalized spaces and take values in Banach spaces.

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