THE FORM OF THE COMPLETE SURFACES OF CONSTANT MEAN CURVATURE

Abdel Radi Abdel Rahman Abdel Gadir Abdel Rahman[1*], Abdelhalim Zaied Elawad Faread[2], Adel AhmedHassan Kubba[3] and and Mohammed Iesa Mohammed Abker[4]

[1,2]Department of Mathematics, Faculty of Education, Omdurman Islamic University, Omdurman, Sudan
[3]Department of Mathematics, Faculty of Education, Nile Valley University, Atbara, Sudan
[4] Department of Mathematics, Faculty of Education, Dalanj University, Dalanj, Sudan

Corresponding Author*
Email id: - dibdelradi78@gmail.com

Abstract

We explained and classified the complete surfaces of constant mean curvature in addition to construct the first examples of complete surface of positive curvature, properly embedded minimal surfaces and we prove that every complete connected immersed surface with positive extrinsic curvature K in $H^2 \times R$ must be properly embedded, homeomorphic to a sphere or a plane. We followed the analytical mathematical method and we found that the complete surface of positive curvature has multi applications in different fields of science specially in physics.

1. Introduction

Differential geometry is a wide domain of modern mathematics, whose significance is increasing at present. One of its origins is in the theory of curves. Everybody who wishes to study geometric problems has to begin by studying the theory curves, where exact definitions, notions, and invariant characteristics are introduced for the first time. Here the initial geometric intuition is formed and then it is developed in the studying of surfaces theory and the geometry of sub manifolds. There exist good and extensive monographs devoted to special curves, but the problems of the general theory are not presented. On the other hand, many interesting and important questions on curves are not discussed, in most cases. We present this is deal with complete smooth surfaces of constant Gaussian curvature k, embedded in the Euclidean space R^3. We treat separately and in the following order the cases $k = 0, k > 0$, and $k < 0$. We show that if the Gaussing curvature is identically zero. The surfaces are union of parallel lines that every complete and connected regular surface of positive and constant Gaussian curvature, is around sphere and, in particular, that is compact surfaces exist only in the positive case. As mentioned, before we can imagine, intuitively a curve as being just a deformation of a straight line without thinking necessary at an analytical representation. We expect the curve to have well defined tangent at each point. This condition should rule out both cusps and self-intersections.

2. The Definition of the Curve

Definition (2.1): A subset $M \subset R^3$ is called a regular curve (or a 1-dimensional smoothsubmanifold of R^3) if, for each point$a \in M$ there is a regular parameterized curve (I, r), whose support, $r(I)$, is an open neighbourhood in M of the point a (i.e. is a set of the form $M \cap U$, where U is an open neighborhood of a inR^3), while the map $r : I \to r(I)$is a homeomorphism, with respect to the topology of subspace of $r(I)$. A parameterizedcurve with these properties is called a local parameterization of the curve Maround the point a. If for a curve M there is a local parameterization (I, r) which is global, i.e., for which $r(I) = M$, the curve is called simple.[7]p(22)

Example (2.2):

1. Any straight line in $R^3$is a simple curve, because it has a global parameterization, given by a function of the form $r : R \to R^3, r(t) = a + b\,t$, where a and $b$are constant vectors, $b \neq 0$.
2. The circular helix is a simple regular curve, with the global parameterization $r : R \to R^3$, given by $r(t)(a\cos t, b\sin t, bt)$.
3. A circle in R^3 is a curve, but it is not simple, since no open interval can be homeomorphic to the circle, which is a compact subset of R^3.

Theorem (2.3): Let $M \subset R^3$be a regular curve and $(I, r = r(t)), (J, \rho = \rho(\tau))$– twolocal parameterizations of M such that $W = r(I) \cap \rho(J) \neq \emptyset$. Then $(r^{-1}(W), r|_{r^{-1}(W)})$and $(\rho^{-1}(W), \rho|_{\rho^{-1}(W)})$are equivalent parameterized curves.

Theorem (2.4): Let $k_1(s), \dots k_{n-1}(s)$ be continuous functions of a parameter $s \in [0,1]$. Assume that $k_1 > 0, \dots k_{n-2} > 0$ there exists a unique up to a rigid motion C^2-regular curve$\gamma \subset E^n$ having the functions k_i as its curvatures and s as the length of arc.[10]p(172)

Remark (2.5): It should be noticed that, usually, the natural parameter along a parameterized curve cannot be expressed in finite terms (i.e., using only elementary functions) with represent to parameter along the curve. This is, impossible even for very simple curves, such that the ellipse

$$\begin{cases} x = acost \\ y = bsint, \end{cases} \tag{2.1}$$

With $a \neq b$, for which the arc length can be expressed only in terms of elliptic functions (this is, actually, the origin of their name!). there for, although the natural parameter is very important for theoretical consideration and for performing the proofs, as the reader will have more than once the opportunity to see in this paper, for concrete examples of parameterized curves we will hardly ever use it.[7]p(22)

3.Manifolds of Constant Curvature:

Definition (3.1): An $n-$dimensional manifold of constant curvature k is a length space x that is locally isometric toM_k^n In order words, for every point $x \in X$ there an $\varepsilon > 0$ and anisometry$\emptyset$form $B(x, \varepsilon)$ on to a ball $B(\emptyset(x), \varepsilon) >\subset M_k^n$.[5]p945)

Theorem (3.2): Let x be a complete, connected, $n-$ dimensional manifold of constant curvature k. When endowed with the induced length metric the universal covering of X is isometric to M_k^n .

Manifold of positive scalar curvature and let S^p denote an embedded $p-$ sphere in X with trivial normal bundle and with $p + q + 1 = n$ and $q \geq 2$. The metric g can be replaced by a pic – metric on$X$ which, on a tubular neighborhood ofS^p **,** is the standard product $d_p^2 + g_{tor}^{q+1}(\delta)$ for some appropriately small δ .

4.Surfaces with Positive Constant Gauss Curvature:
Let us start with a surface $\sum$ endowed with a complete Riemannian metric I of constant Gauss curvature $K(I) = K > 0$. Thus if $\sum$ is simply – connected $\sum$ is isometric to the standard sphere $S^2(r)$ of radius $r = I/\sqrt{K}$, form the cartan – Hadamard theorem

Theorem (4.1): Let $\sum$ be a surface and $\sum \longrightarrow M^3\ (c)$ a c0mplete immersion with positive constant Gauss curvature. Then $f(\sum\ )$ is a totally umbilical round sphere. Observe that, from the Gauss equation, a surface with constant Gauss curvature must also have constant extrinsic curvature. In R^3 both curvature agree and they differ by a constant in H^3 and S^3 moreover, if the gauss curvature is positive then extrinsic curvature of the surface is also positive in H^3 and R^3 however, if the Gauss curvature $K(I)$ is positive in S^3 then the extrinsic curvature is only positive if $K(I) > 1$.[6]p(45)

Theorem (4.2): Let (I, II) be a Codazzi pair on a surface $\sum$ with positive constant extrinsic curvature. Then the (2,0) – part of I with respect to the Riemannian metric II is a holomorphic quadratic form.

5.The tangent plan and **first fundamental form of a ruled surface:**
To compute the coefficients of the first fundamental form of a ruled surface we need, first of all, the partial derivatives of the radius vector of a point of the surface. We have, obviously,

$$r_u^{/} = p^{/} + b\mathrm{b}^{/}; r_v^{/} = \mathrm{b}^{/}_{\mathrm{u}} \qquad (5.1)$$

Thus, the coefficients of the first fundamental form of the surface will be

$$E \equiv r_u^{/} . r_u^{/} = p^{/2} + 2vp^{/}.\mathrm{b}^{/} + v^2\mathrm{b}^{/2} \qquad (5.2)$$

$$f \equiv r_u^{/} . r_v^{/} = p^{/} . \mathrm{b}; \qquad (5.3)$$

$$G \equiv r_v^{/} . r_u^{/} = \mathrm{I} . \qquad (5.4)$$

It follows that the first fundamental form of a ruled surface can be written as:

$$ds^2 = (p^{/2} + 2vp^{/}.\mathrm{b}^{/} + v^2\ \mathrm{b}^{/2})du^2 + 2(p^{/}.\mathrm{b})\ dudv + dv^2 . \qquad (5.5)$$

To find the tangent plane at a point of a ruled surface. We notice. First of all that the direction of the normal to the plan (and. Hence. To the surface) at a given point is given by the vector $r_u^{/} \times r_{v,}^{/}$ i.e., by the vector

$$\mathrm{N} \equiv r_u^{/} \times r_v^{/} = p^{/} \times \mathrm{b} + v\ (\mathrm{b}^{/} \times \mathrm{b}) \qquad (5.6)$$

Therefore. If R is the position vector of a point from the tangent plan to the surface at a point corresponding to the pair of parameters (u , v), then the equation of the tangent plan can be written under the form

(R – r). N = 0,

i.e.
(R – p – vb). ($p^{/} \times$ b) = 0

Or
(R, $p^{/}$, b) + v (R, b$^{/}$, b) – ($p, p^{/}$, b) $-$ v (p, b$^{/}$, b) = 0

Or, also,

$$[\mathrm{R} \times p^{/} + v(\mathrm{R} \times \mathrm{b}^{/}) - p \times p^{/} - v(p \times \mathrm{b}^{/})] = 0. \qquad (5.7)$$

A characteristic property of the ruled surfaces is described by the following proposition. [9] p (187)
Proposition (5.1): The tangent planes to a ruled surface in point located along the same ruling, belong to the pencil of planes determine by that ruling, or, to but if another way, the tangent plane at a point of a ruled surface contains the ruling passing through that point.

Proof: Since the ruling and the tangent plane already have a point in common (the very point of tangency), it is enough to prove that the ruling is parallel to the tangent plane or, which is the same, that it is perpendicular to the normal to the surface at the tangency point. We have, indeed,

N . b = [$p^{/}$ × b + v(b$^{/}$× b)] . b = ($p^{/}$, b, b$^{/}$) + v (b$^{/}$, b, b$^{/}$) = 0.

Definition (5.2): A fundamental forms is the relative to a Cartesian frame in R^3, surfaces $\sum$ can be described implicitly

$$f\ (x\ ,y\ ,z\ ) = 0$$

But for the purposes of differential geometry must be described parametrically

$$r\ (u\ ,v\ ) = \begin{pmatrix} x\ (u\ ,v\ ) \\ y\ (u\ ,v\ ) \\ z\ (u\ ,v\ ) \end{pmatrix}$$

Results:

There are many results and usages of the form of complete surface of constant mean curvature of surface, in different fields of chemistry, physics, engineering and others. In this study we dealt with few examples one of which is used in the manufacture of car tire tubes and the application of curvatures here is used to find the normal curvature and ideal level for its manufacture and, we can model via the computer to know its changes and produced in perfect way. Also, it used for laboratory purposes and for dealing with various liquids that can help researchers to know the liquids nature. We describe and, we can apply the form of complete surface of constant mean curvature of surface and show some applications in Geometry Processing. We also approached the continuous and discrete cases and discussed an implementation. It is an efficient concept that can reduce noise and remove auto-intersections.

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