ON HEYTING ALGEBRA

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Abstract **:** *In mathematics Heyting algebras are special partially ordered sets that constitute a generalisation of Boolean algebra. It is named after Arend Heyting. Heyting algebra arises as models of intuitionistic logic, a logic in which the law of excluded middle does not in general hold. Thus complete Heyting algebra is a central object of study in pointless topology*.

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1. Introduction

A Heyting algebra H is a bounded lattice such that for all a and b in H there is a greatest element x of H having the property

$$a \wedge x \leq b.$$

This element is the relative pseudo-complement of $\neg\, a$ with respect to b and is denoted by $a \rightarrow b$. We write 1 and 0 for the largest and the smallest element of H respectively.

In any Heyting algebra one defines the pseudo-complement $\neg\, x$ of any element x by setting $\urcorner x = x \rightarrow 0$. By definition

$$a \wedge \neg\, a = 0.$$

However, it is not in general true that

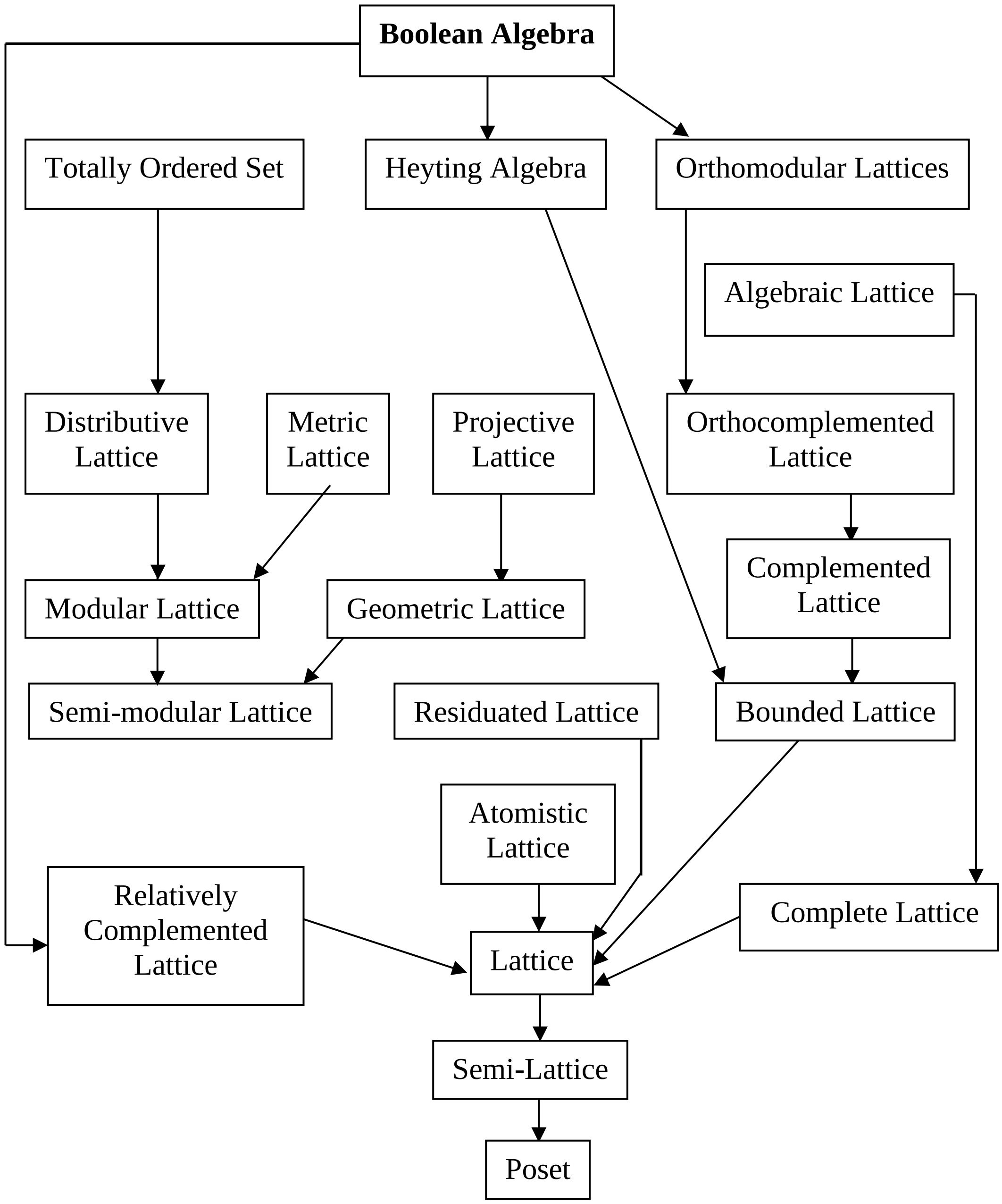
$$a \vee \urcorner a = 1.$$

A complete Heyting algebra is a Heyting algebra that is a complete lattice [1].

A subalgebra of a Heyting algebra H is a subset H_1 of H containing 0 and 1 and closed under the operations $\vee$, $\wedge$ and $\rightarrow$. It follows that it is also closed under $\neg$. A subalgebra is made into a Heyting algebra by the induced operations.

2. Origin and Consequences

The origin of Heyting algebra and its consequences can be easily comprehended from the following diagram :



The above diagram reveals that Heyting algebra has come from the Boolean algebra [2]. Heyting algebras are special partially ordered sets.

3. Definitions

A definition of Heyting algebra can be given by considering the mappings

$$f_a : H \to H \text{ defined by } f_a(x) = a \wedge x$$

for some fixed a in H. A bounded lattice H is a Heyting algebra if and only if all mappings f_a are the lower adjoint of a monotone Galois connection. In this case the respective upper adjoints g_a are given by $g_a(x) = a \to x$, where $\to$ is defined as above.

Another definition of Heyting algebra is as a residuated lattice whose monoid operation is $\wedge$. The monoid unit must then be the top element 1. Commutativity of this monoid implies that the two residuals coincide as $a \to b$.

Given a bounded lattice A with largest and smallest elements 1 and 0 and a binary operation $\to$, these together form a Heyting algebra if and only if the following hold [3] :

1. $a \to a = 1$;
2. $a \wedge (a \to b) = a \wedge b$;
3. $b \wedge (a \to b) = b$;
4. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

4. Distributivity

Heyting algebras are always distributive. Specifically, we always have the identities :

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

The distributive law is sometimes stated as an axiom but in fact it follows from the existence of relative pseudo-complements. The reason is that, being the lower adjoint of a Galois connection, $\wedge$ preserves all existing suprema. Distributivity in turn is just the preservation of binary suprema by $\wedge$.

By a similar argument the following infinite distributive law holds in any complete Heyting algebra [4] :

$$x \wedge \vee Y = \vee \{x \wedge y : y \in Y\}$$

for any element x in H and any subset Y of H. Conversely, any complete lattice satisfying the above infinite distributive law is a complete Heyting algebra with

$$a \to b = \vee \{c : a \wedge c \leq b\}$$

being its relative pseudo-complement operation.

5. Elements

An element x of a Heyting algebra H is called *regular* if either of the following equivalent conditions hold :

1. $x = \neg \neg x$;
2. $x = \neg y$ for some y in H.

The equivalence of these conditions can be restated simply as the identity

$$\neg \neg \neg x = \neg x$$

valid for all x in H.

Elements x and y of a Heyting algebra H are called complements to each other if $x \wedge y = 0$ and $x \vee y = 1$. If it exists, any such y is unique and must, in fact, be equal to $-x$. We call an element x to be *complemented* if it admits a complement.

It is true that if x is complemented, then so is $-x$. And then x and $\neg x$ are complements to each other. However, confusingly, even if x is not complemented, $-x$ may nonetheless have a complement—not equal to x. In any Heyting algebra the elements 0 and 1 are complements to each other.

Any complemented element of a Heyting algebra is regular though the converse is not true in general. In particular, 0 and 1 are always regular.

For any Heyting algebra H, the following conditions are equivalent :

1. H is a Boolean algebra;
2. Every x in H is regular;
3. Every x in H is complemented.

In this case the element $a \to b$ is equal to $\neg a \vee b$.

6. De Morgan's Laws

One of the two De Morgan's laws is satisfied in every Heyting algebra viz.

$$\neg (x \vee y) = \neg x \wedge \neg y \text{ for all } x, y \in H.$$

However, the other De Morgan's law does not always hold. We have instead a weak De Morgan's law :

$$\neg (x \wedge y) = \neg \neg (\neg x \vee \neg y) \text{ for all } x, y \in H.$$

The following statements are equivalent for all Heyting algebras H :

1. H satisfies the De Morgan's law :

   (i) $\neg (x \wedge y) = \neg x \vee \neg y$ for all $x, y \in H$,

   (ii) $\neg (x \wedge y) = \neg x \vee \neg y$ for all regular $x, y \in H$;

2. $\neg \neg (x \vee y) = \neg \neg x \vee \neg y$ for all x, y in H;

3. $\neg \neg (x \vee y) = x \vee y$ for all regular x, y in H;

4. $\neg (\neg x \wedge \neg y) = x \vee y$ for all regular x, y in H;

5. $\neg x \vee \neg \neg x = 1$ for all $x \in H$.

7. Pointless Topology

In mathematics pointless topology is an approach to topology which avoids the mentioning of points [5].

Traditionally a topological space consists of a set of points together with a system of open sets. These open sets with the operations of intersection and union form a lattice with certain properties. Pointless topology then studies lattices like this abstractly without reference to any underlying set of points. Since some of the so-defined lattices do not arise from topological spaces, one may see the category of pointless topological spaces, also called locales, as an extension of the category of ordinary topological spaces.

It is possible to translate most of the concepts of point-set topology into the context of locales and prove analogous theorems. While many important theorems in point-set topology require the axiom of choice, this is not true for some of their analogues in locale theory. This can be useful if one works in a topos which does not have the axiom of choice. A locale is a complete Heyting algebra.

8. Conclusion

Heyting algebras are the example of distributive lattices having at least some members lacking complements. Every element x of a Heyting algebra has, on the other hand, a pseudo-complement denoted as $\neg x$. The pseudo-complement is the greatest element y such that $x \wedge y = 0$. If the pseudo-complement of every element of a Heyting algebra is, in fact, a complement, then the Heyting algebra is a Boolean algebra.

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