Uniqueness of the renormalized solution to quasilinear elliptic equation with Hölder-type dependence and under a local and Fourier boundary conditions

Arouna OUEDRAOGO
Université Norbert ZONGO, Unité de Formation et de Recherche en Sciences et Technologies, Département de Mathématiques
B.P.376 Koudougou, Burkina Faso

Abstract

In this work we prove uniqueness of renormalized solution for elliptic equations of the type div $(A(x,u)\nabla u) = f$ in a bounded set $\Omega \subset \mathbb{R}^N$ with Fourier boundary conditions. The novelty of our results consists in the possibility to deal with cases when $A(x;\, u)$ is only locally Hölder continuous with respect to u and the modulus of Lipschitz continuity is singular.

Key words and phrases: Nonlinear elliptic equations, uniqueness, Hölder nonlinearities, renormalized solutions, Fourier boundary conditions.

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1. Introduction

The present paper is concerned with the uniqueness of the solution to the quasilinear elliptic boundary-value problem on Ω

$$\begin{cases} -div(A(x,u)\nabla u) = f \text{ in } \Omega \\ A(x,u)\nabla u.\eta + \lambda u = g \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

where η is the unit outward normal vector on $\partial\Omega, \Omega$ is a smooth bounded open domain of $\mathbb{R}^N, N \geq 1,\ f \in L^1(\Omega),\ g \in L^1(\partial\Omega),\ \lambda > 0$ and $A(x,s)$ is a Carathéodory function with matrix values.

Classical questions as existence or uniqueness of solutions of (1.1) have to be handled with care. Even the formulation itself of the problem, and the notion of solution considered, deserves some attention, since weak solutions may not have sense. To this purpose, the notion of renormalized solution or entropy solution, introduced in [3],[2] respectively, have proved to be suitable, in particular to deal with the case of coefficients with unbounded growth with respect to u.

When f belongs to $L^2(\Omega)$ the variational solution of (1.1) is unique under a global Lipschitz condition on the function $A(x,s)$ with respect to the variable s (or a global and strong control of the modulus of continuity), see [1, 6] and for more general and nonlinear operator [4, 7]. Moreover in [6, 7] the authors show that if $A(x,s)$ is Hölder continuous in s with a Hölder exponent greater of equal to $1/2$ and if $A(x,s)$ is Lipschitz continuous in x then the solution is unique. For this last result the quasilinear character of the equation and the regularity of $A(x,s)$ in x are crucial. In the present paper we use the framework of renormalized solution (see [8, 10]) which insures the existence of such a solution when f belongs to $L^1(\Omega)$.

Uniqueness results have been recently obtained in [9] in the framework of renormalized solutions and in [5] in the framework of entropy solutions for equations (1.1). In [9], f lies to $L^1(\Omega)$ and $A(x,s)$ is locally Hölder continuous in s with a Hölder exponent greater or equal to $1/2$ and under a global control of the modulus of continuity of $A(x,s)$ with respect to the space variable x. In [5], f lies to $L^1(\Omega) \cap H^{-1}(\Omega)$ and the dependence of $A(x,s)$ with respect to s is not

locally Lipschitz, but authors consider cases when the modulus of Lipschitz continuity is singular.
In the present paper we mix the assumptions and the techniques developed in [5, 9, 11]. We state in Theorem 3.2 that the renormalized solution of (1.1) is unique if $A(x,s)$ is locally Hölder continuous in s with a Hölder exponent greater or equal to $1/2$ and under singularities on the modulus of Lipschitz continuity. The main novelty between our and uniqueness results in [5] is the very local condition on $A(x,s)$ (see assumption (3.7) below). The price to pay to get rid of this global behavior is to assume a regularity with respect to x. Moreover we consider Fourier boundary condtions (see [11]) instead of Dirichlet boundary condtions used in [5].
The paper is organized as follows. In Section 2 we deal with existence and uniqueness of the weak solution of (1.1). Section 3 is devoted to the proof of existence and uniqueness of the renormalized solution of (1.1).

2. Existence and uniqueness of weak solution

We recall the definition of weak solution to problem (1.1).

Definition 2.1. *A measurable function u defined from Ω into $\mathbb{R}$ is called a weak solution of (1.1) if*

$$\begin{cases} u \in H^1(\Omega) \cap L^\infty(\partial\Omega) \text{ such that } f \in L^1(\Omega) \cap [H^1(\Omega)]^*, g \in L^1(\partial\Omega) \cap [H^1(\Omega)]^* \\ \text{and} \\ \int_\Omega A(x,u)\nabla u \nabla\varphi dx + \lambda \int_{\partial\Omega} u\varphi d\sigma = \int_\Omega f\varphi dx + \int_{\partial\Omega} g\varphi d\sigma, \qquad \forall\varphi \in H^1(\Omega). \end{cases} \tag{2.1}$$

Theorem 2.2. *Assume that $A: \Omega \times \mathbb{R} \longrightarrow \mathbb{R}^{N\times N}$ is a Carathéodory function with $A(x,s) = \left(a_{ij}(x,s)\right)_{1\le i,j\le N}$ and such that for every $s, r \in \mathbb{R}$ and a.e. $x \in \Omega$,*

$$\exists\alpha > 0, \exists\beta > 0, \qquad \alpha \le A(x,s) \le \beta, \qquad \forall s \in \mathbb{R}, a.e.\, x \in \Omega; \tag{2.2}$$

for any $r \in \mathbb{R}$ and any $1 \le i,j \le N$, the function $a_{ij}(.,r)$ belongs to $W^{1,\infty}(\Omega)$ and there exists $M > 0$ such that

$$\left|\frac{\partial a_{ij}}{\partial x_k}(x,r)\right| \le M \sum_{1\le i,j\le N} a_{ij}(x,r), \forall s \in \mathbb{R}, \forall 1 \le i,j \le N, a.e.\, x \in \Omega; \tag{2.3}$$

$$\exists H > 0, |A(x,s) - A(x,r)| \le H \frac{|s-r|^{\frac{1}{2}}}{|T_1(s)|^{1-\theta} + |T_1(r)|^{1-\theta}} \tag{2.4}$$

and

$$\frac{1}{2} < \theta \le 1. \tag{2.5}$$

Let $f \in L^1(\Omega) \cap \left[H^1(\Omega)\right]^*$ and $g \in L^1(\partial\Omega) \cap \left[H^1(\Omega)\right]^*$. Then problem (1.1) has a unique weak solution $u \in H^1(\Omega) \cap L^\infty(\partial\Omega)$.

Theorem 2.2 is clearly modeled on the simplest example of Hölder nonlinearity, given by

$$A(x,u) = \alpha(x) + |u|^\theta + |u|^{1/5}, \qquad \frac{1}{2} < \theta \le 1. \tag{2.6}$$

A key point in the proof will be played by the following lemma, which explains condition (2.5).

Lemma 2.3. *(see Lemma 2.1 of [5])*
Let $u \in H^1(\Omega) \cap L^\infty(\partial\Omega)$ be a weak solution of (1.1). Then, for every $\theta \in \left]\frac{1}{2}, 1\right]$ we have

$$\int_{\{x:|u(x)|<1\}} \frac{|\nabla u|^2}{|u|^{2-2\theta}} \leq \frac{\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}}{\alpha(2\theta-1)}. \tag{2.7}$$

Proof. Define $v_\varepsilon = \left[(\varepsilon + |T_1(u)|)^{2\theta-1} - \varepsilon^{2\theta-1}\right] sign(u),\ \varepsilon > 0,$ and use v_ε as test function in (2.1). We have

$$(2\theta-1)\int_\Omega A(x,u)\frac{|\nabla T_1(u)|^2}{(\varepsilon+|T_1(u)|)^{2-2\theta}}\,dx + \int_{\partial\Omega} u[(\varepsilon+|T_1(u)|)^{2\theta-1} - \varepsilon^{2\theta-1}] sign(u)\,d\sigma$$
$$\leq \int_\Omega f v_\varepsilon\,dx + \int_{\partial\Omega} g v_\varepsilon\,d\sigma,$$

which implies, due to (2.2) and since $|v_\varepsilon| \leq (|T_1(u)|)^{2\theta-1} \leq 1$,

$$\alpha(2\theta-1)\int_{\{x:|u(x)|<1\}} \frac{|\nabla u|^2}{(\varepsilon+|u|)^{2-2\theta}}\,dx \leq \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}. \tag{2.8}$$

Letting ε go to zero, thanks to Fatou's Lemma we deduce (2.7). ■

Remark 2.4. *The condition $\theta \in \left]\frac{1}{2}, 1\right]$ is optimal for Lemma 2.3 to hold. It is enough to consider the case of the Laplace operator and $f \geq 0$ to observe that, in the best situation, we have $u \sim \gamma d(x)$ as $x \to \partial\Omega$, where $d(x)$ is the distance function to $\partial\Omega$. This is just consequence of the Hopf boundary lemma, stating in addition that $|\nabla u| \geq \gamma$ at $\partial\Omega$. Therefore, we have, for some $\delta > 0$:*

$$\int_{\{x:|u(x)|<1\}} \frac{|\nabla u|^2}{|u|^{2-2\theta}}\,dx \geq \int_{\{x:d(x)<\delta\}} \frac{|\nabla u|^2}{|u|^{2-2\theta}}\,dx \geq c\int_{\{x:d(x)<\delta\}} \frac{1}{d(x)^{2-2\theta}}\,dx$$

and last integral is not finite for every $\theta \leq \frac{1}{2}$.
We can now prove Theorem 2.2, whose proof follows the ideas of [6] in connection with Lemma 2.3. The main tool is the following lemma which is a truncated version to Theorem 4 in [6].

Lemma 2.5. *Let u, v be two weak solutions of (1.1), we have then for any test function $\varphi \in C^1(\overline{\Omega})$*

$$\int_{\{u-v>0\}} (A(x,u)\nabla u - A(x,v)\nabla v).\nabla\varphi\,dx = 0. \tag{2.9}$$

Proof. Let u, v be two weak solutions of (1.1). We have then, for any test function $\psi \in H^1(\Omega)$:

$$\int_\Omega \big(A(x,u)\nabla u - A(x,v)\nabla v\big).\nabla\psi\, dx + \lambda \int_{\partial\Omega} (u-v)\psi\, d\sigma = 0. \qquad (2.10)$$

Let $\varphi \in C^1(\overline{\Omega})$ with $\varphi \geq 0$ on Ω. We take in (2.10) $\psi = \frac{1}{\varepsilon} T_\varepsilon(u-v)^+\varphi,\ \varepsilon > 0$, and we get

$$\begin{aligned}
&\int_\Omega \big(A(x,u)\nabla u - A(x,v)\nabla v\big).\nabla\varphi \frac{1}{\varepsilon} T_\varepsilon(u-v)^+\, dx + \lambda \int_{\partial\Omega} (u-v)\varphi \frac{1}{\varepsilon} T_\varepsilon(u-v)^+\, d\sigma \\
&= - \int_{\{0<|u-v|<\varepsilon\}} \big(A(x,u)\nabla u - A(x,v)\nabla v\big).\nabla(u-v)^+ \frac{1}{\varepsilon}\varphi\, dx + \lambda \int_{\partial\Omega} (u-v)\varphi \frac{1}{\varepsilon} T_\varepsilon(u-v)^+\, d\sigma \\
&= \int_{\{0<|u-v|<\varepsilon\}} \big(A(x,u) - A(x,v)\big)\nabla u.\nabla(u-v)^+ \frac{1}{\varepsilon}\varphi\, dx \\
&\quad - \int_{\{0<|u-v|<\varepsilon\}} A(x,v)(\nabla u - \nabla v).\nabla(u-v)^+ \frac{1}{\varepsilon}\varphi\, dx \\
&\quad + \lambda \int_{\partial\Omega} (u-v)\varphi \frac{1}{\varepsilon} T_\varepsilon(u-v)^+\, d\sigma \\
&:= A_\varepsilon + B_\varepsilon + C_\varepsilon.
\end{aligned} \qquad (2.11)$$

For the term B_ε, we use (2.2) to obtain

$$B_\varepsilon \leq -\alpha \int_{\{0<|u-v|<\varepsilon\}} |\nabla(u-v)|^2 \frac{1}{\varepsilon}\varphi\, dx. \qquad (2.12)$$

As far as the term A_ε is concerned, Young inequality and relation (2.4) yield

$$\begin{aligned}
A_\varepsilon &\leq \frac{1}{2\alpha} \int_{\{0<|u-v|<\varepsilon\}} \big(A(x,u) - A(x,v)\big)^2 \frac{|\nabla u|^2}{\varepsilon}\varphi\, dx + \frac{\alpha}{2} \int_{\{0<|u-v|<\varepsilon\}} \frac{|\nabla(u-v)|^2}{\varepsilon}\varphi\, dx \\
&\leq \frac{H^2}{2\alpha} \int_{\{0<|u-v|<\varepsilon\}} \frac{|u-v|^{1/2}}{|T_1(u)|^{1-\theta} + |T_1(v)|^{1-\theta}} \frac{|\nabla u|^2}{\varepsilon}\varphi\, dx + \frac{\alpha}{2} \int_{\{0<|u-v|<\varepsilon\}} \frac{|\nabla(u-v)|^2}{\varepsilon}\varphi\, dx \qquad (2.13) \\
&\leq \frac{H^2}{2\alpha} \int_{\{0<|u-v|<\varepsilon\}} \frac{|\nabla u|^2}{|T_1(u)|^{2-2\theta}}\varphi\, dx + \frac{\alpha}{2} \int_{\{0<|u-v|<\varepsilon\}} \frac{|\nabla(u-v)|^2}{\varepsilon}\varphi\, dx.
\end{aligned}$$

From (2.12) and (2.13), we deduce that

$$A_\varepsilon + B_\varepsilon \leq \frac{H^2}{2\alpha} \int_{\{0<|u-v|<\varepsilon\}} \frac{|\nabla u|^2}{|T_1(u)|^{2-2\theta}}\varphi\, dx. \qquad (2.14)$$

Remark that

$$\bigcap_{\varepsilon>0} \{x : 0 < |u(x) - v(x)| < \varepsilon\} = \{x : 0 < |u(x) - v(x)| \leq 0\} = \emptyset.$$

The decreasing continuity of the measure implies that

$$\text{meas}\{x: 0 < |u(x) - v(x)| < \varepsilon\} \longrightarrow 0, \text{as } \varepsilon \longrightarrow 0.$$

Moreover, since

$$\frac{|\nabla u|^2}{|T_1(u)|^{2-2\theta}} \leq \frac{|\nabla u|^2}{|u|^{2-2\theta}} \chi_{\{|u|<1\}} + |\nabla u|^2,$$

from Lemma 2.3, the fact that u belongs to $H^1(\Omega)$ and φ is regular, it follows that $\frac{|\nabla u|^2}{|T_1(u)|^{2-2\theta}}\varphi \in L^1(\Omega)$. Therefore, we have

$$\lim_{\varepsilon\to 0}\int_{\{0<|u-v|<\varepsilon\}} \frac{|\nabla u|^2}{|T_1(u)|^{2-2\theta}}\varphi\, dx = 0. \qquad (2.15)$$

For the term C_ε of (2.11), we have

$$\lim_{\varepsilon\to 0} C_\varepsilon = \lambda\int_{\partial\Omega}(u-v)^+\,\varphi\, d\sigma \geq 0. \qquad (2.16)$$

Passing to the limit as $\varepsilon \longrightarrow 0$ in (2.11) and taking account (2.10), (2.14)-(2.16), we deduce

$$\int_{\{u-v>0\}} \big(A(x,u)\nabla u - A(x,v)\nabla v\big).\nabla\varphi\, dx \leq 0. \qquad (2.17)$$

Taking $M-\varphi$ in place of φ in (2.17), with M a constant sufficiently large so that $M-\varphi \geq 0$, gives

$$\int_{\{u-v>0\}} \big(A(x,u)\nabla u - A(x,v)\nabla v\big).\nabla\varphi\, dx \geq 0. \qquad (2.18)$$

At last (2.17) and (2.18) allow to conclude that (2.9) holds true. The proof of Lemma 2.5 is then complete. ■

With the help of Lemma 2.5 we now turn to Theorem 2.2.
Proof of Theorem 2.2. We use Lemma 2.5 with $\varphi(x) = exp(c\sum_{i=1}^N x_i)$, where $c > 0$.
Let us define

$$\tilde{a}_{i,j}(x,r) = \int_0^r a_{i,j}(x,s)ds.$$

Assumption (2.3) implies that both $\tilde{a}_{i,j}(x,u)$ and $\tilde{a}_{i,j}(x,v)$ belong to $H^1(\Omega)$ and for $r = u, v$,

$$\frac{\partial\tilde{a}_{i,j}(x,r)}{\partial x_k} = a_{ij}(x,r)\frac{\partial r}{\partial x_k} + \int_0^r \frac{\partial a_{i,j}(x,s)}{\partial x_k}\,ds. \qquad (2.19)$$

Since $\frac{\partial\varphi}{\partial x_k} = c\varphi$, using (2.19), we have

$$\int_{\{u-v>0\}} \big(A(x,u)\nabla u - A(x,v)\nabla v\big).\nabla\varphi\, dx$$
$$= c\int_{\{u-v>0\}}\sum_{1\leq i,j\leq N}\Big(\frac{\partial\tilde{a}_{i,j}(x,u)}{\partial x_j} - \frac{\partial\tilde{a}_{i,j}(x,v)}{\partial x_j}\Big)\varphi\, dx$$
$$+c\int_{\{u-v>0\}}\sum_{1\leq i,j\leq N}\int_u^v \frac{\partial a_{i,j}(x,s)}{\partial x_j}\,ds.$$

On the other hand, passing to the limit as $\varepsilon \longrightarrow 0$ in (2.11) and taking account (2.10) and (2.16), we deduce,

$$\lambda\int_{\partial\Omega}(u-v)^+\varphi d\sigma = 0, \qquad \forall\varphi \in C^1(\overline{\Omega})\ with\ \varphi \geq 0.$$

Then, we get $(u-v)^+ = 0$ a.e. on $\partial\Omega$. In the sequel, let us define $w = (u-v)^+$ which belongs to $L^\infty(\Omega) \cap H_0^1(\Omega)$ and is such that $u = w + v$ almost everywhere on $\{u - v > 0\}$. Moreover, $\tilde{a}_{i,j}(x, v+w) - \tilde{a}_{i,j}(x, v)$ lies in $L^\infty(\Omega) \cap H_0^1(\Omega)$. So, a few computations and the integration by parts formula give

$$\begin{aligned}
&\int_{\{u-v>0\}} \big(A(x,u)\nabla u - A(x,v)\nabla v\big).\nabla\varphi\,dx \\
&= c\int_\Omega \sum_{1\le i,j\le N} \Big(\frac{\partial \tilde{a}_{i,j}(x,v+w)}{\partial x_j} - \frac{\partial \tilde{a}_{i,j}(x,v)}{\partial x_j}\Big)\varphi\,dx \\
&\quad + c\int_\Omega \sum_{1\le i,j\le N} \int_{v+w}^{v} \frac{\partial a_{i,j}(x,s)}{\partial x_j}\,ds \\
&= -c^2 \int_\Omega \sum_{1\le i,j\le N} \big(\tilde{a}_{i,j}(x,v+w) - \tilde{a}_{i,j}(x,v)\big)\varphi\,dx \\
&\quad + c\int_\Omega \sum_{1\le i,j\le N} \int_{v+w}^{v} \frac{\partial a_{i,j}(x,s)}{\partial x_j}\,ds \\
&= -c\int_\Omega \int_v^{v+w} \Big(c\sum_{1\le i,j\le N} a_{ij}(x,s) + \sum_{1\le i,j\le N} \frac{\partial a_{i,j}(x,s)}{\partial x_j}\Big)\,ds\,\varphi\,dx.
\end{aligned}$$

Because $\varphi \geq 0$ in Ω, from assumptions (2.2) and (2.3), we obtain for c sufficiently large ($c > 2N^2M$ for example) that

$$\begin{aligned}
&\int_{\{u-v>0\}} \big(A(x,u)\nabla u - A(x,v)\nabla v\big).\nabla\varphi\,dx \\
&\le \frac{-\alpha c}{2}\int_\Omega \int_v^{v+w} ds\,\varphi\,dx \qquad (2.20)\\
&\le \frac{-\alpha c}{2}\int_\Omega w\,dx \text{ a.e. in } \Omega,
\end{aligned}$$

where $w = (u-v)^+$.
Finally from (2.9), (2.20) and Fatou's lemma, it follows that

$$\int_\Omega w dx \le 0,$$

which leads to a contradiction unless $w \equiv 0$.
The proof of Theorem 2.2 is complete. ■

3. Uniqueness of renormalized solution

In this section we generalize the example of the previous section, which is of course very special to many regards, in particular $A(x,s)$ was supposed to be singular at only one point. The boundedness assumption on $A(x,s)$ was also not essential but for considering standard weak solutions. Here we assume that $A(x,s)$ only satisfies

$$\exists\alpha > 0, \qquad A(x,s) \ge \alpha, \qquad \forall s \in \mathbb{R}, a.e.\, x \in \Omega \tag{3.1}$$

and that

$$\forall K > 0, \exists C_K > 0, \qquad \sup_{|s|\le K} A(x,s) \le C_K, a.e.\, x \in \Omega \tag{3.2}$$

For any $k > 0$ we denote by T_k the truncation function at height $k, T_k(s) = \max(-k, \min(k, s))$ for any $s \in \mathbb{R}$ and we define the continuous function h_n by

$$h_n(s) = 1\left|\frac{T_{2n}(s) - T_n(s)}{n}\right|. \qquad (3.3)$$

In order to deal with possibly unbounded function $A(x, s)$, a generalized concept of solution is needed. Following [8] (see also [10]) we recall the definition of a renormalized solution of (1.1).

Definition 3.1. *A measurable function $u: \Omega \to \mathbb{R}$ (u is finite almost everywhere in Ω) is called a renormalized solution of (1.1) if*

$$T_k(u) \in H^1(\Omega) \cap L^1(\partial\Omega) \text{ for any } k \geq 0, \qquad (3.4)$$

if for any function $h \in W^{1,\infty}(\mathbb{R})$ with compact support and $h(0) = 0$, u satisfies the equation

$$\begin{aligned}\int_\Omega h(u)A(x,u)\nabla u \cdot \nabla \psi \, dx + \int_\Omega h'(u)A(x,u)\nabla u.\nabla u \, \psi \, dx \\ +\lambda \int_{\partial\Omega} u\, h(u)\, \psi \, d\sigma = \int_\Omega f\, h(u)\, \psi \, dx + \int_{\partial\Omega} g\, h(u)\, \psi \, d\sigma,\end{aligned} \qquad (3.5)$$

for all $\psi \in L^\infty(\Omega) \cap H_0^1(\Omega)$ and

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{n<|u|<2n\}} A(x,u)\nabla u.\nabla u \, dx = 0. \qquad (3.6)$$

Let us now generalize assumption (2.4). First of all, we set for any $K > 0$

$$\omega_{\varepsilon,K}(s) := \sup\left\{\frac{|A(x,s) - A(x,r)|}{|s-r|^{1/2}}, (x,r) : \varepsilon < |s-r| < 2\varepsilon,\ |s| \leq K,\ |r| \leq K,\ x \in \Omega\right\}.$$

Note that, thanks to assumption (3.2), $\omega_{\varepsilon,K}(s)$ is a locally bounded function. We assume that $\omega_{\varepsilon,K}(s)$ satisfies

$$\exists \bar{\omega}_K \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}), \qquad \omega_{\varepsilon,K}(s)^2 \leq \bar{\omega}_K(s) \qquad \forall s \in \mathbb{R} \text{ with } |s| \leq K,\ \forall \varepsilon \leq \varepsilon_0, \quad (3.7)$$

for some $\varepsilon_0 > 0$.

Our main result is the following.

Theorem 3.2. *Assume (2.3), (3.1)-(3.2) and (3.7) hold. Let $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$. Then, the renormalized solution of (1.1) is unique.*

To prove Theorem 3.2 we mix the methods developed by Chipot and Carrillo in [6] and Guibé in [9]. The main tool is the following lemma.

Lemma 3.3. *For any φ belonging to $C^1(\overline{\Omega})$*

$$\lim_{n\to+\infty}\int_{\{u-v>0\}}\big(h_n(u)A(x,u)\nabla u - h_n(v)A(x,v)\nabla v\big).\nabla\varphi\,dx = 0. \tag{3.8}$$

Proof. Let φ *belonging to* $C^1(\overline{\Omega})$ with $\varphi \geq 0$ on Ω and let n be a positive integer. We consider for $\varepsilon \leq \varepsilon_0$, the test function $W_\varepsilon = \frac{1}{\varepsilon}T_{2\varepsilon}(u-v)^+\varphi$ which belongs to $L^\infty(\Omega)\cap H_0^1(\Omega)$ due to (3.4) and the regularity of φ.
Choosing $h = h_n$ in (3.5) written in u yields

$$\int_\Omega h_n(u)A(x,u)\nabla u.\nabla W_\varepsilon\,dx + \int_\Omega h'_n(u)A(x,u)\nabla u.\nabla u\,W_\varepsilon\,dx$$
$$+\lambda\int_{\partial\Omega} u\,h_n(u)\,W_\varepsilon\,d\sigma = \int_\Omega f\,h_n(u)\,W_\varepsilon\,dx + \int_{\partial\Omega} g\,h_n(u)\,W_\varepsilon\,d\sigma \tag{3.9}$$

which can be rewritten as

$$\int_\Omega h_n(u)A(x,u)\nabla u.\nabla\varphi\frac{T_{2\varepsilon}(u-v)^+}{\varepsilon}\,dx + \int_{\{0<u-v<2\varepsilon\}} h_n(u)A(x,u)\nabla u.\nabla\Big(\frac{(u-v)^+}{\varepsilon}\Big)\varphi\,dx$$
$$+\int_\Omega h'_n(u)A(x,u)\nabla u.\nabla u\frac{T_{2\varepsilon}(u-v)^+}{\varepsilon}\varphi\,dx + \lambda\int_{\partial\Omega} u\,h_n(u)\,W_\varepsilon\,d\sigma$$
$$= \int_\Omega h_n(u)fW_\varepsilon\,dx + \int_{\partial\Omega} g\,h_n(u)\,W_\varepsilon\,d\sigma.$$

Subtracting the equivalent equality written in v gives

$$\int_\Omega\big(h_n(u)A(x,u)\nabla u - h_n(v)A(x,v)\nabla v\big).\nabla\varphi\frac{T_{2\varepsilon}(u-v)^+}{\varepsilon}\,dx$$
$$+\int_{\{0<u-v<2\varepsilon\}}\big(h_n(u)A(x,u)\nabla u - h_n(v)A(x,v)\nabla v\big).\nabla\Big(\frac{(u-v)^+}{\varepsilon}\Big)\varphi\,dx$$
$$+\int_\Omega\big(h'_n(u)A(x,u)\nabla u.\nabla u - h'_n(v)A(x,v)\nabla v.\nabla v\big)\frac{T_{2\varepsilon}(u-v)^+}{\varepsilon}\varphi\,dx$$
$$+\lambda\int_\Omega\big(h_n(u)\,u - h_n(v)\,v\big)\,W_\varepsilon\,dx$$
$$= \int_\Omega\big(h_n(u)-h_n(v)\big)fW_\varepsilon\,dx + \int_{\partial\Omega}\big(h_n(u)-h_n(v)\big)gW_\varepsilon\,dx,$$

which reads as
$A_{n,\varepsilon} + B_{n,\varepsilon} + C_{n,\varepsilon} + D_{n,\varepsilon} = E_{n,\varepsilon} + F_{n,\varepsilon}$. (3.10)

Observe that, since $f \in L^1(\Omega)$ *and* $g \in L^1(\partial\Omega)$ the fact that $|W_\varepsilon| \leq 2\varphi$ uniformly with respect to ε, the regularity of φ and since $h_n \to 1$ in L^∞ weak$-^*$ and almost everywhere in Ω as n goes to infinity, the Lebesgue dominated convergence theorem implies that

$$\begin{cases} \limsup_{\varepsilon\to 0}\lim_{n\to+\infty}|D_{n,\varepsilon}| = 2\int_{\partial\Omega}(u-v)^+\varphi \geq 0, \\ \limsup_{\varepsilon\to 0}\lim_{n\to+\infty}|E_{n,\varepsilon}| = 0, \\ \limsup_{\varepsilon\to 0}\lim_{n\to+\infty}|F_{n,\varepsilon}| = 0. \end{cases} \tag{3.11}$$

We split $B_{n,\varepsilon}$ into

$$B_{n,\varepsilon} = B^1_{n,\varepsilon} + B^2_{n,\varepsilon} + B^3_{n,\varepsilon}, \tag{3.12}$$

with

$$B^1_{n,\varepsilon} = \int_{\{0<u-v<2\varepsilon\}} h_n(u)A(x,u)(\nabla u - \nabla v).\nabla\Big(\frac{(u-v)^+}{\varepsilon}\Big)\varphi\,dx$$

$$B^2_{n,\varepsilon} = -\int_{\{0<u-v<2\varepsilon\}} h_n(u)\big(A(x,u) - A(x,v)\big)\nabla v.\nabla\Big(\frac{(u-v)^+}{\varepsilon}\Big)\varphi\,dx$$

$$B^3_{n,\varepsilon} = \int_{\{0<u-v<2\varepsilon\}} \big(h_n(u) - h_n(v)\big)A(x,v)\nabla v.\nabla\Big(\frac{(u-v)^+}{\varepsilon}\Big)\varphi\,dx.$$

Using (3.1) we get

$$B^1_{n,\varepsilon} \geq \alpha \int_{\{0<u-v<2\varepsilon\}} h_n(u)\frac{1}{\varepsilon}|\nabla(u-v)|^2\varphi\,dx. \tag{3.13}$$

Young inequality and relation (3.7) yield

$$\begin{aligned} -B^2_{n,\varepsilon} &\leq \frac{1}{2\alpha}\int_{\{0<u-v<2\varepsilon\}} |A(x,u) - A(x,v)|^2|\nabla v|^2\frac{1}{\varepsilon}\varphi\,dx \\ &\quad + \frac{\alpha}{2}\int_{\{0<u-v<2\varepsilon\}} h_n(u)\frac{1}{\varepsilon}|\nabla(u-v)|^2\varphi\,dx \\ &\leq \frac{1}{\alpha}\int_{\{0<u-v<2\varepsilon\}} \bar{\omega}_K(u)|\nabla v|^2\varphi\,dx + \frac{\alpha}{2}\int_{\{0<u-v<2\varepsilon\}} h_n(u)\frac{1}{\varepsilon}|\nabla(u-v)|^2\varphi\,dx. \end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14), we get

$$B^1_{n,\varepsilon} + B^2_{n,\varepsilon} \geq \frac{\alpha}{2} \int_{\{0<u-v<2\varepsilon\}} h_n(u)\frac{1}{\varepsilon}|\nabla(u-v)|^2\varphi\,dx - \frac{1}{\alpha}\int_{\{0<u-v<2\varepsilon\}} \omega_K(u)|\nabla v|^2\varphi\,dx \geq -\frac{1}{\alpha}\int_{\{0<u-v<2\varepsilon\}} \bar{\omega}_K(u)|\nabla v|^2\varphi\,dx. \quad (3.15)$$

As v belongs to $H^1(\Omega), \varphi \in C^1(\overline{\Omega})$ and $\bar{\omega}_K \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$, it follows that $\bar{\omega}_K(u)|\nabla v|^2\varphi \in L^1(\Omega)$. Therefore, passing to the limit as $\varepsilon \longrightarrow 0$ in (3.15) yields

$$\liminf_{\varepsilon\to 0}\left(B^1_{n,\varepsilon} + B^2_{n,\varepsilon}\right) \geq 0. \quad (3.16)$$

As far as $B^3_{n,\varepsilon}$ is concerned, we have

$$\begin{aligned} |h_n(u) - h_n(v)| &= \left|\left|\frac{T_{2n}(v) - T_n(v)}{n}\right| - \left|\frac{T_{2n}(u) - T_n(u)}{n}\right|\right| \\ &\leq \left|\frac{T_{2n}(v) - T_{2n}(u) + T_n(u) - T_n(v)}{n}\right| \\ &\leq \frac{2}{n}|u-v|. \end{aligned}$$

Therefore,

$$\begin{aligned} \left|B^3_{n,\varepsilon}\right| &\leq \frac{1}{\varepsilon}\int_{\{0<u-v<2\varepsilon\}} |h_n(u) - h_n(v)|\left|A(x,v)\nabla v.\nabla\left((u-v)^+\right)\varphi\right| dx \\ &\leq \frac{2}{n\varepsilon}\int_{\{0<u-v<2\varepsilon\}} |u-v|\left|A(x,v)\nabla v.\nabla\left((u-v)^+\right)\varphi\right| dx \\ &\leq \frac{4}{n}\int_{\{0<u-v<2\varepsilon\}} \left|A(x,v)\nabla v.\nabla\left((u-v)^+\right)\varphi\right| dx. \end{aligned} \quad (3.17)$$

As v belongs to $H^1(\Omega)$, $(u-v)^+ \in H^1_0(\Omega)$, $\varphi \in C^1(\overline{\Omega})$ and taking account (3.2), it follows that

$A(x,v)\nabla v.\nabla((u-v)^+)\varphi \in L^1(\Omega)$.

So, passing to the limit as $\varepsilon \longrightarrow 0$ in (3.17) yields

$$\liminf_{\varepsilon\to 0} B^3_{n,\varepsilon} = 0. \quad (3.18)$$

Therefore, combining (3.16) and (3.18), we can pass to the limit in (3.12) as ε tends to 0, and then as n tends to $+\infty$ to get

$$\lim_{n\to+\infty} \liminf_{\varepsilon\to 0} B_{n,\varepsilon} \geq 0. \tag{3.19}$$

Now, we deal with the term $C_{n,\varepsilon}$ of (3.10).

Due to the fact that $|h'_n(s)| \leq \frac{1}{n}, T_{2\varepsilon} \leq 2\varepsilon$ and supp $h'_n = [-2n, -n] \cup [n, 2n]$, we have

$$\left|\int_\Omega h'_n(u)A(x,u)\nabla u.\nabla u \frac{T_{2\varepsilon}(u-v)^+}{\varepsilon}\varphi\, dx\right| \leq \frac{2\|\varphi\|_{L^\infty(\Omega)}}{n} \int_{\{n<|u|<2n\}} A(x,u)\nabla u.\nabla u\, dx.$$

Therefore, condition (3.6) allows us to obtain

$$\lim_{n\to+\infty} \limsup_{\varepsilon\to 0} |C_{n,\varepsilon}| = 0. \tag{3.20}$$

With arguments already used we know that

$$\lim_{\varepsilon\to 0} A_{n,\varepsilon} = \int_{\{u-v>0\}} \big(h_n(u)A(x,u)\nabla u - h_n(v)A(x,v)\nabla v\big).\nabla\varphi\, dx. \tag{3.21}$$

From equality (3.10) together with (3.11), (3.19)-(3.20), it follows that

$$\limsup_{n\to+\infty} \int_{\{u-v>0\}} \big(h_n(u)A(x,u)\nabla u - h_n(v)A(x,v)\nabla v\big).\nabla\varphi\, dx \leq 0. \tag{3.22}$$

Taking $M-\varphi$ in place of φ in (3.22), with M sufficiently large so that $M-\varphi \geq 0$, gives

$$\liminf_{n\to+\infty} \int_{\{u-v>0\}} \big(h_n(u)A(x,u)\nabla u - h_n(v)A(x,v)\nabla v\big).\nabla\varphi\, dx \geq 0. \tag{3.23}$$

At last (3.22) and (3.23) allow to conclude that (3.8) holds true. The proof of Lemma 3.3 is complete. ∎

With the help of Lemma 3.3 we now prove Theorem 3.2 by proceding as in [9].
Proof of Theorem 3.2. We use Lemma 3.3 with $\varphi(x) = exp(c\sum_{i=1}^{N} x_i)$, where $c > 0$. Since $h_n(s) = 0, \forall\, |s| \geq 2n$, we have

$$h_n(u)A(x,u)\nabla u\chi_{\{u-v>0\}} = h_n(T_{2n}(u))A(x,T_{2n}(u))\nabla T_{2n}(u)\chi_{\{T_{2n}(u)-T_{2n}(v)>0\}}$$

almost everywhere in Ω. To shorten the notations we denote by u^{2n} the field $T_{2n}(u)$ and by v^{2n} the field $T_{2n}(v)$. It follows that (3.8) can be rewritten as

$$\lim_{n\to+\infty}\int_{\{u^{2n}-v^{2n}>0\}}\big(h_n(u^{2n})A(x,u^{2n})\nabla u^{2n}-h_n(v^{2n})A(x,v^{2n})\nabla v^{2n}\big).\nabla\varphi\,dx=0. \quad (3.24)$$

Let us define

$$\tilde{a}^n_{i,j}(x,r)=\int_0^r a_{i,j}(x,s)h_n(s)\,ds.$$

Due to the regularity (3.4) of $T_k(u)$ and $T_k(v)$, assumption (2.3) implies that both $\tilde{a}^n_{i,j}(x,u^{2n})$ and $\tilde{a}^n_{i,j}(x,v^{2n})$ belong to $H^1(\Omega)$ and for $r=u^{2n},v^{2n}$,

$$\frac{\partial\tilde{a}^n_{i,j}(x,r)}{\partial x_k}=h_n(r)a_{ij}(x,r)\frac{\partial r}{\partial x_k}+\int_0^r h_n(s)\frac{\partial a_{i,j}(x,s)}{\partial x_k}\,ds. \quad (3.25)$$

Since $\frac{\partial\varphi}{\partial x_k}=c\varphi$, using (3.25), we have

$$\begin{aligned}\int_{\{u^{2n}-v^{2n}>0\}}&\big(h_n(u^{2n})A(x,u^{2n})\nabla u^{2n}-h_n(v^{2n})A(x,v^{2n})\nabla v^{2n}\big).\nabla\varphi\,dx\\ &=c\int_{\{u^{2n}-v^{2n}>0\}}\sum_{1\le i,j\le N}\Big(\frac{\partial\tilde{a}_{i,j}(x,u^{2n})}{\partial x_j}-\frac{\partial\tilde{a}_{i,j}(x,v^{2n})}{\partial x_j}\Big)\varphi\,dx\\ &\quad+c\int_{\{u^{2n}-v^{2n}>0\}}\sum_{1\le i,j\le N}\int_{u^{2n}}^{v^{2n}}h_n(s)\frac{\partial a_{i,j}(x,s)}{\partial x_j}\,ds.\end{aligned}$$

Let us define $w_{2n}=(u^{2n}-v^{2n})^+$ which belongs to $L^\infty(\Omega)\cap H^1_0(\Omega)$ and is such that $u^{2n}=w_{2n}+v^{2n}$ almost everywhere on $\{u^{2n}-v^{2n}>0\}$. Since $\tilde{a}^n_{i,j}(x,w_{2n}+v^{2n})-\tilde{a}^n_{i,j}(x,v^{2n})$ lies in $L^\infty(\Omega)\cap H^1_0(\Omega)$, a few computations and the integration by parts formula give

$$\int_{\{u^{2n}-v^{2n}>0\}} \big(h_n(u^{2n})A(x,u^{2n})\nabla u^{2n} - h_n(v^{2n})A(x,v^{2n})\nabla v^{2n}\big).\nabla\varphi\, dx$$

$$= c\int_\Omega \sum_{1\le i,j\le N} \Big(\frac{\partial \tilde{a}^n_{i,j}(x,v^{2n}+w_{2n})}{\partial x_j} - \frac{\partial \tilde{a}^n_{i,j}(x,v^{2n})}{\partial x_j}\Big)\varphi\, dx$$

$$+c\int_\Omega \sum_{1\le i,j\le N} \int_{v^{2n}+w_{2n}}^{v^{2n}} h_n(s)\frac{\partial a_{i,j}(x,s)}{\partial x_j}\, ds$$

$$= -c^2\int_\Omega \sum_{1\le i,j\le N} \big(\tilde{a}^n_{i,j}(x,v^{2n}+w_{2n}) - \tilde{a}^n_{i,j}(x,v^{2n})\big)\varphi\, dx$$

$$+c\int_\Omega \sum_{1\le i,j\le N} \int_{v^{2n}+w_{2n}}^{v^{2n}} h_n(s)\frac{\partial a_{i,j}(x,s)}{\partial x_j}\, ds$$

$$= -c\int_\Omega \int_{v^{2n}}^{v^{2n}+w_{2n}} h_n(s)\Big(c\sum_{1\le i,j\le N} a_{ij}(x,s) + \sum_{1\le i,j\le N}\frac{\partial a_{i,j}(x,s)}{\partial x_j}\Big)\, ds\,\varphi\, dx.$$

Because $\varphi \geq 0$ in Ω, from assumptions (2.3) and (3.1), we obtain for c sufficiently large ($c > 2N^2M$ for example) that

$$\int_{\{u^{2n}-v^{2n}>0\}} \big(h_n(u^{2n})A(x,u^{2n})\nabla u^{2n} - h_n(v^{2n})A(x,v^{2n})\nabla v^{2n}\big).\nabla\varphi\, dx \le \frac{-\alpha c}{2}\int_\Omega \int_{v^{2n}}^{v^{2n}+w_{2n}} h_n(s)ds\,\varphi\, dx. \tag{3.26}$$

Since u and v are finite almost everywhere in h_n while converges to 1 almost everywhere in $\mathbb{R}$ and is bounded by 1 we obtain

$$\lim_{n\to+\infty}\int_{v^{2n}}^{v^{2n}+w_{2n}} h_n(s)ds = \int_v^{v+w} ds = w \qquad \text{almost everywhere in } \Omega, \tag{3.27}$$

where $w = (u-v)^+$.
Finally from (3.24), (3.26), (3.27) and Fatou's lemma, it follows that

$$\int_\Omega w dx \le 0,$$

which leads to a contradiction unless $w \equiv 0$.
The proof of Theorem 3.2 is complete. ■

References

[1] M. Artola, *Sur une classe de problèmes paraboliques quasi-linéaires*, Boll. Un. Mat. Ital. B(6), 5(1):5170, 1986.

[2] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre & J.L. Vazquez, *An L^1 theory of existence and uniqueness of nonlinear elliptic equations*, Ann. Scuola. Norm. Sup. Pisa, 22: 241-273, 1995.

[3] L. Boccardo, I. Diaz, D. Giachetti & F. Murat, *Existence of a solution for a weaker form of a nonlinear elliptic equation*, Recent advances in nonlinear elliptic and parabolic problems (Nancy, 1988). Pitman Res. Notes Math. Ser. 208, 229-246, Longman, 1989.

[4] L. Boccardo, T. Gallouët & F. Murat, *Unicité de la solution de certaines équations elliptiques non Linéaires*, C. R. Acad. Sci. Paris Sér. I Math., 315(11):1159-1164, 1992.

[5] L. Boccardo & A. Porretta, *Uniqueness for elliptic problems with Hölder-type dependence on the Solution*, Commun. Pure Appl. Anal. 12, No.4, 1569-1585 (2013).

[6] J. Carrillo & M. Chipot, *On some nonlinear elliptic equations involving derivatives of the nonlinearity*, Proc. Roy. Soc. Edinburgh Sect. A, 100(3-4):281-294, 1985.

[7] J. Carrillo & G. Michaille, *Uniqueness results and monotonicity properties for strongly nonlinear elliptic variational inequalities*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 16(1):137-166, 1989.

[8] G. Dal Maso, F. Murat, L. Orsina & A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 28(4):741-808, 1999.

[9] O. Guibé, *Uniqueness of the solution to quasilinear elliptic equations under a local condition on the diffusion matrix*, Adv. Math. Sci. Appl. 17, No.2, 357-368 (2007).

[10] F. Murat, *Equations elliptiques non linéaires avec second membre L^1 ou mesure*, In Compte Rendus du 26ième congrès d'Analyse Numérique, les Karellis, 1994.

[11] I. Nyanquini & S. Ouaro; *Entropy solution for nonlinear elliptic problem involving variable exponent and Fourier type boundary condition*, Afr. Mat. 23, No 2, 205-228 (2012).