CUBIC WEAK BI-IDEALS OF Γ-NEAR-RINGS

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ABSTRACT. In this paper, we introduced the new notion of cubic weak bi-ideals of Γ-near-rings, which is the generalized concept of cubic weak bi-ideals of near-rings. We also investigated some of its properties with examples.

1. INTRODUCTION

Zadeh [21] initiated the concept of fuzzy sets in 1965. Near-ring theory was introduced by Pilz [14]. Gamma-near-ring was introduced by Satyanarayana [15] in 1984. The concept of bi-ideals was applied to near rings and Gamma-near-rings [17, 18]. Kim et al.[6] defined the concept of fuzzy R-subgroups of near-rings. The idea of fuzzy ideals of near-rings was first proposed by Kim et al.[5]. Moreover, Manikantan [7] introduced the notion of fuzzy bi-ideals of near-rings and discussed some of its properties. Yong Uk Cho et al.[20] introduced the concept of weak bi-ideals applied to near-rings. Chinnadurai et al.[4] introduced fuzzy weak bi-ideals of near-rings. Thillaigovindan et al.[19] introduced interval valued fuzzy ideals of near rings. Jun et al.[10] introduced the concept of cubic subgroups. This structure encompasses interval-valued fuzzy set and fuzzy set. Also Jun et al.[12] introduced the notion of cubic ideals of semigroups. Chinnadurai et al.[3] introduced the notion of cubic weak bi-ideals of near-rings. In this paper, we defined a new notion of cubic weak bi-ideals of Γ-near-rings, we also discussed some of its properties with examples.

2. PRELIMINARIES

In this section, we listed some basic definitions related to cubic weak bi-ideals of Γ-near-rings. Throughout this paper R denotes a left Γ-near-ring.

Definition 2.1. [1] A near-ring is an algebraic system $(R, +, \cdot)$ consisting of a non-empty set R together with two binary operations called $+$ and $\cdot$ such that $(R, +)$ is a group not necessarily abelian and $(R, \cdot)$ is a semigroup connected by the following distributive law: $x\cdot(y + z) = x \cdot y + x \cdot z$ valid for all $x, y, z \in R$. We use the word 'near-ring' to means 'left near-ring'. We denote xy insted of $x\cdot y$. An ideal I of a near-ring R is a subset of R such that (i) (I,+) is a normal subgroup of (R,+) (ii) $RI \subseteq I$ (iii) $(x + a)y - xy \in I$ for any $a \in I$ and $x, y \in R$. A R-subgroup H of a near-ring R is the subset of R such that (i) (H,+) is a subgroup of (R, +) (ii) $RH \subseteq H$ (iii) $HR \subseteq H$.

Note that H is a left R-subgroup of R if H satisfies (i) and (ii) and a right R-subgroup of R if H satisfies (i) and (iii).

Key words and phrases. Near-rings, Γ-near-rings, weak bi-ideals, fuzzy weak bi-ideals, cubic weak bi-ideals, homomorphisam of cubic weak bi-ideals.

Definition 2.2. [7] Let R be a near-ring. Given two subsets A and B of R, we define the following products $AB = \{ab \mid a \in A, b \in B\}$ and $A\star B = \{(a'+b)a - a'a \mid a, a' \in A, b \in B\}$.

Definition 2.3. [17] A subgroup B of $(R, +)$ is said to be bi-ideal of R if $BRB \cap B \star RB \subseteq B$.

Definition 2.4. [20] A subgroup B of $(R, +)$ is said to be weak bi-ideal of R if $BBB \subseteq B$.

Definition 2.5. [16] Let $(M, +)$ be a group and Γ-be a non-empty set. Then M is said to be Γ-near-ring, if there exists a mapping $M \times \Gamma \times M \to M$ (The image of (x, α, y) is denoted by $x\alpha y$) satisfying the following conditions
1. $(x + y)\alpha z = x\alpha z + y\alpha z$,
2. $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition 2.6. [16] Let M be a Γ-near-ring. A normal subgroup $(I, +)$ of $(M, +)$ is called
1. a left ideal if $x\alpha(y + i) - x\alpha y \in I$,
2. a right ideal if $i\alpha x \in I$ for all $x, y \in M, \alpha \in \Gamma, i \in I$,
3. an ideal if it is both a left ideal and a right ideal of M.

A $\Gamma-$near-ring M is said to be zero-symmetric if $a\alpha 0 = 0$ for all $a \in M$, $\alpha \in \Gamma$, where 0 is the additive identity in M.

Definition 2.7. [18] A subgroup B of (M,+) is a bi-ideal iff $B\Gamma M\Gamma B \subseteq B$.

Definition 2.8. [15] Let M be a Γ-near-ring. Given two subsets A and B of M, we define the following products $A\Gamma B = \{a\alpha b \mid a \in A, b \in B$ and $\alpha \in \Gamma\}$ and also define another operation on $\star$ on the class of subset M is defined by $A\Gamma \star B = \{(a' + b)\gamma a - a'\gamma a \mid a, a' \in A, b \in B$ and $\gamma \in \Gamma$ $\}$.

Definition 2.9. [2] A fuzzy subset μ of a set X is a function $\mu : X \to [0, 1]$.

Definition 2.10. [2] Let μ and λ be any two fuzzy subsets of R. Then $\mu\lambda$ is fuzzy subset of R defined by

$$(\mu\lambda)(x) = \begin{cases} \sup\limits_{x=yz} \min\{\mu(y), \lambda(z)\} & \text{if } x = yz \text{ for all } x, y, z \in R \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.11. [7] A fuzzy subgroup μ of $(R, +)$ is said to be fuzzy bi-ideal of R if $\mu R\mu \cap \mu \star R\mu \subseteq \mu$

Definition 2.12. [1] Let R be a near-ring and μ be a fuzzy subset of R. We say μ is a fuzzy subnear-ring of R if
1. $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$
2. $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in R$.

Definition 2.13. [1] Let R be a near-ring and μ be a fuzzy subset of R. Then μ is called a fuzzy ideal of R, if
1. $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$
2. $\mu(y + x - y) \geq \mu(x)$
3. $\mu(xy) \geq \mu(y)$
4. $\mu((x + z)y - xy) \geq \mu(z)$ for all $x, y \in R$.

A fuzzy subset with (1) to (3) is called a fuzzy left ideal of R, whereas a fuzzy subset with (1),(2) and (4) are called a fuzzy right ideal of R.

Definition 2.14. [1] A fuzzy subset μ of a near-ring R is called a fuzzy R-subgroup of R if
1. μ is a fuzzy subgroup of $(R, +)$
2. $\mu(xy) \geq \mu(y)$
3. $\mu(xy) \geq \mu(x)$ for all $x, y \in R$.
A fuzzy subset with (1) and (2) is called a fuzzy left R-subgroup of R, whereas a fuzzy subset with (1) and (3) is called a fuzzy right R-subgroup of R.

Definition 2.15. [4] A fuzzy subgroup μ of R is called fuzzy weak bi-ideal of R, if

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}.$$

Definition 2.16. [2] Let X be a non-empty set. A mapping $\overline{\mu} : X \rightarrow D[0,1]$ is called an interval-valued (in short i-v) fuzzy subset of X, if for all $x \in X, \overline{\mu}(x) = [\mu^-(x), \mu^+(x)]$, where μ^- and μ^+ are fuzzy subsets of X such that $\mu^-(x) \leq \mu^+(x)$. Thus $\overline{\mu}(x)$ is an interval (a closed subset of [0,1]) and not a number from the interval [0,1] as in the case of fuzzy set.

Definition 2.17. [3] A cubic set $A = \langle \overline{\mu}, \omega \rangle$ of R is called cubic subgroup of R, if
1. $\overline{\mu}(x - y) \geq \min\{\overline{\mu}(x), \overline{\mu}(y)\}$
2. $\omega(x - y) \leq \max\{\omega(x), \omega(y)\}$ $\forall x, y \in R$.

Definition 2.18. [3] A cubic subgroup $A = \langle \overline{\mu}, \omega \rangle$ of R is called cubic weak bi-ideal of R, if
1. $\overline{\mu}(xyz) \geq \min\{\overline{\mu}(x), \overline{\mu}(y), \overline{\mu}(z)\}$
2. $\omega(xyz) \leq \max\{\omega(x), \omega(y), \omega(z)\}$ $\forall x, y, z \in R$.

Definition 2.19. [3] Let A_i be cubic weak bi-ideals of near-rings R_i for $i = 1, 2, 3, \ldots, n$. Then the cubic direct product of $A_i (i = 1, 2, \ldots, n)$ is a function $\bar{\mu}_1 \times \bar{\mu}_2 \times \cdots \times \bar{\mu}_n : R_1 \times R_2 \times \cdots \times R_n \rightarrow D[0,1]$,
$\omega_1 \times \omega_2 \times \cdots \times \omega_n : R_1 \times R_2 \times \cdots \times R_n \rightarrow [0,1]$ defined by
$(\bar{\mu}_1 \times \bar{\mu}_2 \cdots \times \bar{\mu}_n)(x_1, x_2, \ldots, x_n) = \min\{\bar{\mu}_1(x_1), \bar{\mu}_2(x_2), \ldots, \bar{\mu}_n(x_n)\}$ and
$(\omega_1 \times \omega_2 \times \cdots \times \omega_n)(x_1, x_2, \ldots, x_n) = \max\{\omega_1(x_1), \omega_2(x_2), \ldots, \omega_n(x_n)\}$.

Definition 2.20. [3] Let $A_1 = \langle \bar{\mu}_1, \omega_1 \rangle$ *and* $A_2 = \langle \bar{\mu}_2, \omega_2 \rangle$ be any two cubic subsets of R. Then $A_1 A_2$ is cubic subsets of R defined by:

$$(A_1A_2)(x) = \begin{cases} (\bar{\mu}_1\bar{\mu}_2)(x) = \begin{cases} \sup\limits_{x=yz} \min\{\overline{\mu}_1(y), \overline{\mu}_2(z)\} & \text{if } x = yz \text{ for all } x, y, z \in R \\ [0,0] & \text{otherwise} \end{cases} \\ (\omega_1\omega_2)(x) = \begin{cases} \inf\limits_{x=yz} \max\{\omega_1(y), \omega_2(z)\} & \text{if } x = yz \text{ for all } x, y, z \in R \\ 1 & \text{otherwise} \end{cases} \end{cases}$$

Definition 2.21. [9] Let f be a mapping from a set X to Y and $A = \langle \overline{\eta}, \lambda \rangle$ be a cubic set of X then the image of X (i.e.,) $C_f(A) = \langle C_f(\overline{\eta}), C_f(\lambda) \rangle$ is a cubic set of Y defined by

$$C_f(A)(y) = \begin{cases} C_f(\bar{\mu})(y) = \begin{cases} \sup\limits_{f(x)=y} \overline{\eta}(x) & \text{if } f^{-1}(y) \neq \emptyset \\ [0,0] & \text{otherwise} \end{cases} \\ C_f(\lambda)(y) = \begin{cases} \inf\limits_{f(x)=y} \lambda(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases} \end{cases}$$

and let f be a mapping from a set X to Y and $A = \langle \overline{\eta}, \lambda \rangle$ is a cubic set of Y, then the pre image of Y (i.e.,) ${C_f}^{-1}(A) = \left\langle {C_f}^{-1}(\overline{\eta}), {C_f}^{-1}(\lambda) \right\rangle$ is a cubic set of X is defined by

$$
{C_f}^{-1}(A)(x) = \begin{cases} {C_f}^{-1}(\overline{\mu})(x) = \overline{\eta}(f(x)) \\ {C_f}^{-1}(\lambda)(x) = \lambda(f(x)) \end{cases}
$$

3. Cubic weak bi-ideals of Γ-near-rings

In this section, we introduced the new notion of cubic weak bi-ideals of Γ-near-rings and discuss some of its properties.

Definition 3.1. A cubic set $A = \langle \overline{\eta}, \omega \rangle$ of Γ-near-ring R is called cubic subgroup of R, if

(i) $\overline{\eta}(x - y) \geq \min\{\overline{\eta}(x), \overline{\eta}(y)\}$
(ii) $\omega(x - y) \leq \max\{\omega(x), \omega(y)\}\ \forall x, y \in R$.

Definition 3.2. A cubic subgroup $A = \langle \overline{\mu}, \omega \rangle$ of a Γ-near-ring R is called cubic weak bi-ideal of Γ-near-ring R, if

(i) $\overline{\eta}(x\alpha y\beta z) \geq \min\{\overline{\eta}(x), \overline{\eta}(y), \overline{\eta}(z)\}$
(ii) $\omega(x\alpha y\beta z) \leq \max\{\omega(x), \omega(y), \omega(z)\}\ \forall x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

Example 3.3. Let $R = \{a, b, c, d\}$ be a non-empty set with binary operation $+$ and $\Gamma = \{\alpha, \beta\}$ be a non-empty set of binary operations as shown in the following tables:

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

α	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	a	a	c	c
d	b	b	d	d

β	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	b	d	c
d	b	b	c	d

Clearly (i) $(R, +)$ is a group (ii) $x\alpha(y + z) = x\alpha y + x\alpha z$ (iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for every $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. Then R is a Γ-near-ring.
Define a cubic set $A = (\overline{\eta}, \omega)$ in R as follows:

N	$\overline{\eta}(x)$	$\omega(x)$
a	[0.8.0.9]	0.2
b	[0.6,0.7]	0.4
c	[0.2,0.3]	0.6
d	[0.2,0.3]	0.8

Hence, $A = (\overline{\eta}, \omega)$ is a cubic weak bi-ideal of Γ-near-ring R.

Definition 3.4. Let $A_1 = \langle \overline{\eta}_1, \omega_1 \rangle\ and\ A_2 = \langle \overline{\eta}_2, \omega_2 \rangle$ be any two cubic subsets of R. Then $A_1 A_2$ is a cubic subsets of R defined by:

$$
(A_1A_2)(x) = \begin{cases} (\overline{\eta}_1\overline{\eta}_2)(x) = \begin{cases} \sup\limits_{x = y\alpha z} \min\{\overline{\eta}_1(y), \overline{\eta}_2(z)\} & \text{for all } x, y, z \in R \text{ and } \alpha \in \Gamma \\ [0, 0] & \text{otherwise} \end{cases} \\ (\omega_1\omega_2)(x) = \begin{cases} \inf\limits_{x = y\alpha z} \max\{\omega_1(y), \omega_2(z)\} & \text{for all } x, y, z \in R \text{ and } \alpha \in \Gamma \\ 1 & \text{otherwise} \end{cases} \end{cases}
$$

Theorem 3.5. *Let $A = \langle \overline{\eta}, \omega \rangle$ be a cubic subgroup of Γ-near-ring R. Then $A = \langle \overline{\eta}, \omega \rangle$ is a cubic weak bi-ideal of Γ-near-ring $R \Leftrightarrow AAA \sqsubseteq A$.*
(i.e., $\overline{\eta}\ \overline{\eta}\ \overline{\eta} \subseteq \overline{\eta}$ and $\omega\ \omega\ \omega \supseteq \omega$)

Proof. Assume that $A = \langle \overline{\eta}, \omega \rangle$ is a cubic weak bi-ideal of Γ-near-ring R.
Let $x, y, z, p, q \in R$ and $\alpha, \beta \in \Gamma$, such that $x = y\alpha z$ and $y = p\beta q$. Then

$$\begin{aligned}
(\overline{\eta}\ \overline{\eta}\ \overline{\eta})(x) &= \sup_{x=y\alpha z} \{\min\{(\overline{\eta}\ \overline{\eta})(y), \overline{\eta}(z)\}\} \\
&= \sup_{x=y\alpha z} \left\{ \min \left\{ \sup_{y=p\beta q} \min\{\overline{\eta}(p), \overline{\eta}(q)\}, \overline{\eta}(z) \right\} \right\} \\
&= \sup_{x=y\alpha z} \sup_{y=p\beta q} \{\min\{\min\{\overline{\eta}(p), \overline{\eta}(q)\}, \overline{\eta}(z)\}\} \\
&= \sup_{x=p\beta q\alpha z} \{\min\{\overline{\eta}(p), \overline{\eta}(q), \overline{\eta}(z)\}\} \\
&\leq \sup_{x=p\beta q\alpha z} \overline{\eta}(p\beta q\alpha z) \\
&= \overline{\eta}(x)
\end{aligned}$$

If x can not be expressed as $x = y\alpha z$ then $(\overline{\eta}\ \overline{\eta}\ \overline{\eta})(x) = \overline{0} \leq \overline{\eta}(x)$.
In both cases $\overline{\eta}\ \overline{\eta}\ \overline{\eta} \subseteq \overline{\eta}$.

$$\begin{aligned}
(\omega\ \omega\ \omega)(x) &= \inf_{x=y\alpha z} \{\max\{(\omega\ \omega)(y), \omega(z)\}\} \\
&= \inf_{x=y\alpha z} \left\{ \max \left\{ \inf_{y=p\beta q} \max\{\omega(p), \omega(q)\}, \omega(z) \right\} \right\} \\
&= \inf_{x=y\alpha z} \inf_{y=p\beta q} \{\max\{\max\{\omega(p), \omega(q)\}, \omega(z)\}\} \\
&= \inf_{x=p\beta q\alpha z} \{\max\{\omega(p), \omega(q), \omega(z)\}\} \\
&\geq \inf_{x=p\beta q\alpha z} \omega(p\beta q\alpha z) \\
&= \omega(x)
\end{aligned}$$

If x can not be expressed as $x = y\alpha z$ then $(\omega\ \omega\ \omega)(x) = 1 \geq \omega(x)$.

In both cases $\omega\ \omega\ \omega \supseteq \omega$.

Hence $AAA \sqsubseteq A$.

Conversly, assume that $AAA \sqsubseteq A$ holds. To prove that $A = \langle \overline{\eta}, \omega \rangle$ is a cubic weak bi-ideal of Γ-near-ring R.
For any $x, y, z, a \in R$ and $\alpha, \alpha_1, \beta, \beta_1$ such that $a = x\alpha y\beta z$ then

$$\begin{aligned}
\overline{\eta}(x\alpha y\beta z) &= \overline{\eta}(a) \geq (\overline{\eta}\ \overline{\eta}\ \overline{\eta})(a) \\
&= \sup_{a=b\alpha_1 c} \min\{(\overline{\eta}\ \overline{\eta})(b),\ \overline{\eta}(c)\} \\
&= \sup_{a=b\alpha_1 c} \left\{ \min \left\{ \sup_{b=p\beta_1 q} \min\{\overline{\eta}(p), \overline{\eta}(q)\}, \overline{\eta}(c) \right\} \right\} \\
&= \sup_{a=p\beta_1 q\alpha_1 c} \{\min\{\overline{\eta}(p), \overline{\eta}(q)\}, \overline{\eta}(c)\}\} \\
\overline{\eta}(x\alpha y\beta z) &\geq \min\{\overline{\eta}(x),\ \overline{\eta}(y),\ \overline{\eta}(z)\} \\
\omega(x\alpha y\beta z) = \omega(a) &\leq (\omega\ \omega\ \omega)(a) \\
&= \inf_{a=b\alpha_1 c} \max\{(\omega\omega)(b), \omega(c)\}
\end{aligned}$$

$$= \inf_{a=b\alpha_1 c} \left\{ \max \left\{ \inf_{b=p\beta_1 q} \max\{\omega(p), \omega(q)\}, \omega(c) \right\} \right\}$$
$$= \inf_{a=p\beta_1 q\alpha_1 c} \{\max\{\omega(p), \omega(q), \omega(c)\}\}$$
$$\omega(x\alpha y\beta z) \leq \max\{\omega(x), \omega(y), \omega(z)\}$$

Hence $A = \langle \overline{\eta}, \omega \rangle$ is a cubic weak bi-ideal of Γ-near-ring R. □

Theorem 3.6. *Let A_1 and A_2 be two cubic weak bi-ideals of Γ-near-ring R then the product A_1A_2 is a cubic weak bi-ideal of Γ-near-ring R.*

Proof. Let $A_1 = \langle \overline{\eta}_1, \omega_1 \rangle$ and $A_2 = \langle \overline{\eta}_2, \omega_2 \rangle$ be two cubic weak bi-deals of Γ-near-ring R.

Since $\overline{\eta}_1$ and $\overline{\eta}_2$ are interval-valued fuzzy weak bi-ideals of Γ-near-ring R then

$$\begin{aligned}(\overline{\eta}_1\overline{\eta}_2)(x-y) &= \sup_{x-y=p\alpha q} \min\{\overline{\eta}_1(p), \overline{\eta}_2(q)\} \\ &\geq \sup_{x-y=p_1\alpha_1 q_1 - p_2\alpha_2 q_2 \leq (p_1-p_2)(q_1-q_2)} \min\{\overline{\eta}_1(p_1-p_2), \overline{\eta}_2(q_1-q_2)\} \\ &\geq \sup \min\{\min\{\overline{\eta}_1(p_1), \overline{\eta}_1(p_2)\}, \min\{\overline{\eta}_2(q_1), \overline{\eta}_2(q_2)\}\} \\ &= \sup \min\{\min\{\overline{\eta}_1(p_1), \overline{\eta}_2(q_1)\}, \min\{\overline{\eta}_1(p_2), \overline{\eta}_2(q_2)\} \\ &= \min\left\{ \sup_{x=p_1\alpha_1 q_1} \min\left\{\overline{\eta}_1(p_1), \overline{\eta}_2(q_1)\}, \sup_{y=p_2\alpha_2 q_2} \min\{\overline{\eta}_1(p_2), \overline{\eta}_2(q_2)\right\}\right\} \\ &= \min\{(\overline{\eta}_1\overline{\eta}_2)(x), (\overline{\eta}_1\overline{\eta}_2)(y)\}\end{aligned}$$

It follows that $(\overline{\eta}_1\overline{\eta}_2)$ is an interval-valued fuzzy subgroup of Γ-near-ring R. Further

$$\begin{aligned}(\overline{\eta}_1\overline{\eta}_2)(\overline{\eta}_1\overline{\eta}_2)(\overline{\eta}_1\overline{\eta}_2) &= \overline{\eta}_1\overline{\eta}_2(\overline{\eta}_1\overline{\eta}_2\overline{\eta}_1)\overline{\eta}_2 \\ &\subseteq \overline{\eta}_1\overline{\eta}_2(\overline{\eta}_2\overline{\eta}_2\overline{\eta}_2)\overline{\eta}_2 \\ &\subseteq \overline{\eta}_1(\overline{\eta}_2\overline{\eta}_2\overline{\eta}_2) \\ &\subseteq (\overline{\eta}_1\overline{\eta}_2)\end{aligned}$$

Therefore $(\overline{\eta}_1\overline{\eta}_2)$ is an interval-valued fuzzy weak bi-ideals of Γ-near-ring R.
Since ω_1, ω_2 are fuzzy weak bi-ideals of Γ-near-ring R, then

$$\begin{aligned}(\omega_1\omega_2)(x-y) &= \inf_{x-y=p\alpha q} \max\{\omega_1(p), \omega_2(q)\} \\ &\leq \inf_{x-y=p_1\alpha_1 q_1 - p_2\alpha_2 q_2 \leq (p_1-p_2)(q_1-q_2)} \max\{\omega_1(p_1-p_2), \omega_2(q_1-q_2)\} \\ &\leq \inf \max\{\max\{\omega_1(p_1), \omega_1(p_2)\}, \max\{\omega_2(q_1), \omega_2(q_2)\}\} \\ &= \inf \max\{\max\{\omega_1(p_1), \omega_2(q_1)\}, \max\{\omega_1(p_2), \omega_2(q_2)\} \\ &= \max\left\{ \inf_{x=p_1\alpha_1 q_1} \max\left\{\omega_1(p_1), \omega_2(q_1)\}, \inf_{y=p_2\alpha_2 q_2} \max\{\omega_1(p_2), \omega_2(q_2)\right\}\right\} \\ &= \max\{(\omega_1\omega_2)(x), (\omega_1\omega_2)(y)\}\end{aligned}$$

It follows that $(\omega_1\omega_2)$ is a fuzzy subgroup of Γ-near-ring R. Further

$$\begin{aligned}(\omega_1\omega_2)(\omega_1\omega_2)(\omega_1\omega_2) &= \omega_1\omega_2(\omega_1\omega_2\omega_1)\omega_2 \\ &\supseteq \omega_1\omega_2(\omega_2\omega_2\omega_2)\omega_2 \\ &\supseteq \omega_1(\omega_2\omega_2\omega_2) \\ &\supseteq (\omega_1\omega_2)\end{aligned}$$

Thus, $(\omega_1\omega_2)$ is a fuzzy weak bi-ideals of Γ-near-ring R.
Hence, $A_1A_2 = \langle(\overline{\eta}_1\overline{\eta}_2), (\omega_1\omega_2)\rangle$ is a cubic weak bi-ideal of Γ-near-ring R. □

Remark 3.7. Let A_1 and A_2 be two cubic weak bi-ideals of Γ-near-ring R then the product A_2A_1 is also a cubic weak bi-ideal of Γ-near-ring R.

Theorem 3.8. *Let $A = \langle\overline{\eta}, \omega\rangle$ be a cubic weak bi-ideal of Γ-near-ring R, then the set $R_A = \{x \in R \mid A(x) = A(0)\}$ (i.e., $R_A = \{x \in R \mid \overline{\eta}(x) = \overline{\eta}(0)$ and $\omega(x) = \omega(0)\}$) is a weak bi-ideal of Γ-near-ring R.*

Proof. Let $A = \langle\overline{\eta}, \omega\rangle$ be a cubic weak bi-ideal of R. Let $x, y \in R_A$.
Then $A(x) = A(0)$ and $A(y) = A(0)$. (i.e.,) $\overline{\eta}(x) = \overline{\eta}(0), \omega(x) = \omega(0)$ and $\overline{\eta}(y) = \overline{\eta}(0), \omega(y) = \omega(0)$. Since $\overline{\eta}$ is an interval-valued fuzzy weak bi-ideal of Γ-near-ring R. We have $\overline{\eta}(x) = \overline{\eta}(0)$ and $\overline{\eta}(y) = \overline{\eta}(0)$. Then
$\overline{\eta}(x-y) \geq \min\{\overline{\eta}(x), \overline{\eta}(y)\} = \min\{\overline{\eta}(0), \overline{\eta}(0)\} = \overline{\eta}(0)$ and ω is a fuzzy weak bi-ideal of Γ-near-ring R, we have $\omega(x) = \omega(0)$ and $\omega(y) = \omega(0)$ then
$\omega(x-y) \leq \max\{\omega(x), \omega(y)\} = \max\{\omega(0), \omega(0)\} = \omega(0)$. Thus $x - y \in R_A$
For every $x, y, z \in R_A$ and $\alpha, \beta \in \Gamma$. Then $A(x) = A(0)$, $A(y) = A(0)$ and $A(z) = A(0)$. Since $\overline{\eta}$ is an interval-valued fuzzy weak bi-ideal of Γ-near-ring R, we have $\overline{\eta}(x) = \overline{\eta}(0)$, $\overline{\eta}(y) = \overline{\eta}(0)$ and $\overline{\eta}(z) = \overline{\eta}(0)$ then
$\overline{\eta}(x\alpha y\beta z) \geq \min\{\overline{\eta}(x), \overline{\eta}(y), \overline{\eta}(z)\} = \min\{\overline{\eta}(0), \overline{\eta}(0), \overline{\eta}(0)\} = \overline{\eta}(0)$ and
ω is a fuzzy weak bi-ideal of Γ-near-ring R, we have $\omega(x) = \omega(0), \omega(y) = \omega(0)$, $\omega(z) = \omega(0)$ and $\omega(x\alpha y\beta z) \leq \max\{\omega(x), \omega(y), \omega(z)\} = \max\{\omega(0), \omega(0), \omega(0)\}$
$= \omega(0)$. Thus $x\alpha y\beta z \in R_A$.
Hence R_A is a weak bi-ideal of Γ-near-ring R. □

Theorem 3.9. *Let $\{A_i\}_{i\in\Omega} = \langle\overline{\eta}_i, \omega_i : i \in \Omega\rangle$ be a family of cubic weak bi-ideals Γ-near-ring R, then $\bigcap_{i\in\Omega} A_i = \left\langle \bigcap_{i\in\Omega} \overline{\eta}_i, \bigcup_{i\in\Omega} \omega_i \right\rangle$ is also a family of cubic weak bi-ideal Γ-near-ring R, where Ω is any index set.*

Proof. Let $\{A_i\}_{i\in\Omega} = \langle\overline{\eta}_i, \omega_i : i \in \Omega\rangle$ be a family of cubic weak bi-ideals of Γ-near-ring R.
Let $x, y, z \in R$, $\alpha, \beta \in \Gamma$ and $\bigcap_{i\in\Omega} \overline{\eta}_i(x) = (\inf_{i\in\Omega} \overline{\eta}_i)(x) = \inf_{i\in\Omega} \overline{\eta}_i(x)$,
$\bigcup_{i\in\Omega} \omega_i(x) = (\sup_{i\in\Omega} \omega_i)(x) = \sup_{i\in\Omega} \omega_i(x)$

Since $\overline{\eta}_i$ is a family of interval-valued fuzzy weak bi-ideals of Γ-near-ring R, we have

$$\begin{aligned}
\bigcap_{i\in\Omega} \overline{\eta}_i(x-y) &= \inf_{i\in\Omega} \overline{\eta}_i(x-y) \\
&\geq \inf_{i\in\Omega} \min\{\overline{\eta}_i(x), \overline{\eta}_i(y)\} \\
&= \min\left\{\inf_{i\in\Omega} \overline{\eta}_i(x), \inf_{i\in\Omega} \overline{\eta}_i(y)\right\} \\
&= \min\left\{\bigcap_{i\in\Omega} \overline{\eta}_i(x), \bigcap_{i\in\Omega} \overline{\eta}_i(y)\right\}
\end{aligned}$$

and ω_i is a family of fuzzy weak bi-ideals of Γ-near-ring R. We have

$$\begin{aligned}\bigcup_{i\in\Omega}\omega_i(x-y) &= \sup_{i\in\Omega}\omega_i(x-y)\\ &\leq \sup_{i\in\Omega}\max\{\omega_i(x),\omega_i(y)\}\\ &= \max\left\{\sup_{i\in\Omega}\omega_i(x),\sup_{i\in\Omega}\omega_i(y)\right\}\\ &= \max\left\{\bigcup_{i\in\Omega}\omega_i(x),\bigcup_{i\in\Omega}\omega_i(y)\right\}\end{aligned}$$

Thus, $\bigcap_{i\in\Omega} A_i$ is a cubic subgroup of Γ-near-ring R.

Again,

$$\begin{aligned}\bigcap_{i\in\Omega}\overline{\eta}_i(x\alpha y\beta z) &= \inf_{i\in\Omega}\overline{\eta}_i(x\alpha y\beta z)\\ &\geq \inf_{i\in\Omega}\min\{\overline{\eta}_i(x),\overline{\eta}_i(y),\overline{\eta}_i(z)\}\\ &= \min\left\{\inf_{i\in\Omega}\overline{\eta}_i(x),\inf_{i\in\Omega}\overline{\eta}_i(y),\inf_{i\in\Omega}\overline{\eta}_i(z)\right\}\\ &= \min\left\{\bigcap_{i\in\Omega}\overline{\eta}_i(x),\bigcap_{i\in\Omega}\overline{\eta}_i(y),\bigcap_{i\in\Omega}\overline{\eta}_i(z)\right\}\\ \bigcup_{i\in\Omega}\omega_i(x\alpha y\beta z) &= \sup_{i\in\Omega}\omega_i(x\alpha y\beta z)\\ &\leq \sup_{i\in\Omega}\max\{\omega_i(x),\omega_i(y),\omega_i(z)\}\\ &= \max\left\{\sup_{i\in\Omega}\omega_i(x),\sup_{i\in\Omega}\omega_i(y),\sup_{i\in\Omega}\omega_i(z)\right\}\\ &= \max\left\{\bigcup_{i\in\Omega}\omega_i(x),\bigcup_{i\in\Omega}\omega_i(y),\bigcup_{i\in\Omega}\omega_i(z)\right\}\end{aligned}$$

Hence, $\bigcap_{i\in\Omega} A_i = \left\langle \bigcap_{i\in\Omega}\overline{\eta}_i, \bigcup_{i\in\Omega}\omega_i \right\rangle$ is a family of cubic weak bi-ideal of Γ-near-ring R. □

Theorem 3.10. *Let H be a non empty subset of Γ-near-ring R and $A=\langle\overline{\eta},\omega\rangle$ be a cubic subset of Γ-near-ring R defined by*

$$A(x)=\begin{cases}\overline{\eta}(x)=\begin{cases}[p_1,p_2] & \text{if } x\in H\\ [q_1,q_2] & \text{otherwise}\end{cases}\\ \omega(x)=\begin{cases}1-p & \text{if } x\in H\\ 1-q & \text{otherwise}\end{cases}\end{cases}$$

for all $x\in R, [p_1,p_2],[q_1,q_2]\in D[0,1]$ and $p,q\in[0,1]$ with $[p_1,p_2]>[q_1,q_2], p>q$. Then H is a weak bi-ideal of Γ-near-ring $R \Leftrightarrow A=\langle\overline{\eta},\omega\rangle$ is a cubic weak bi-ideal of Γ-near-ring R.

Proof. Assume that H is a weak bi-ideal of Γ-near-ring R. Let $x,y\in H$ we consider four cases:

International Journal For Research In Mathematics And Statistics ISSN: 208-2662

(1) $x \in H$ and $y \in H$
(2) $x \in H$ and $y \notin H$
(3) $x \notin H$ and $y \in H$
(4) $x \notin H$ and $y \notin H$

Case (i) If $x \in H$ and $y \in H$. Then $\overline{\eta}(x) = [p_1, p_2] = \overline{\eta}(y)$ and $\omega(x) = 1 - p$ $= \omega(y)$. Since H is a weak bi-deal Γ-near-ring R, then $x - y \in R$. Thus $\overline{\eta}(x - y) = [p_1, p_2] = \min\{[p_1, p_2], [p_1, p_2]\} = \min\{\overline{\eta}(x), \overline{\eta}(y)\}$ and $\omega(x - y) = 1 - p = \max\{1 - p, 1 - p\} = \max\{\omega(x), \omega(y)\}$.

Case (ii) If $x \in H$ and $y \notin H$. Then $\overline{\eta}(x) = [p_1, p_2], \overline{\eta}(y) = [q_1, q_2]$ and $\omega(x) = 1-p, \omega(y) = 1-q$. Clearly, $\overline{\eta}(x-y) \geq \min\{\overline{\eta}(x), \overline{\eta}(y)\} = \min\{[p_1, p_2], [q_1, q_2]\}$ $= [q_1, q_2]$ and $\omega(x - y) \leq \max\{\omega(x), \omega(y)\} = \max\{1 - p, 1 - q\} = 1 - q$.
Now, $\overline{\eta}(x - y) = [p_1, p_2]$ *or* $[q_1, q_2]$ according as $x - y \in H$ or $x - y \notin H$. By assumption that $[p_1, p_2] > [q_1, q_2]$ and $p > q$, we have $\overline{\eta}(x - y) \geq \min\{\overline{\eta}(x), \overline{\eta}(y)\}$ and $\omega(x - y) \leq \max\{\omega(x), \omega(y)\}$.

Similarly we can prove that case (iii).

Case (iv) If $x \notin H$ and $y \notin H$. Then $\overline{\eta}(x) = [q_1, q_2] = \overline{\eta}(y)$ and $\omega(x) = 1 - q = \omega(y)$. So, $\min\{\overline{\eta}(x), \overline{\eta}(y)\} = [q_1, q_2]$ and $\max\{\omega(x), \omega(y)\} = 1 - q$. Next, $\overline{\eta}(x - y) = [p_1, P_2]$ or $[q_1, q_2]$ and $\omega(x - y) = 1 - p$ or $1 - q$, according as $x - y \in H$ or $x - y \notin H$. So, $A = \langle \overline{\eta}, \omega \rangle$ is a cubic subgroup of R. Now, let $x, y, z \in H$. We have the following eight cases:

(1) $x \in H, y \in H$ and $z \in H$
(2) $x \notin H, y \in H$ and $z \in H$
(3) $x \in H, y \notin H$ and $z \in H$
(4) $x \in H, y \in H$ and $z \notin H$
(5) $x \notin H, y \notin H$ and $z \in H$
(6) $x \in H, y \notin H$ and $z \notin H$
(7) $x \notin H, y \in H$ and $z \notin H$
(8) $x \notin H, y \notin H$ and $z \notin H$

These cases can be proved by similar arguments of the cubic cases above.
Hence, $\overline{\eta}(x\alpha y\beta z) \geq \min\{\overline{\eta}(x), \overline{\eta}(y), \overline{\eta}(z)\}$ and $\omega(x\alpha y\beta z) \leq \max\{\omega(x), \omega(y), \omega(z)\}$.
Therefore, $A = \langle \overline{\eta}, \omega \rangle$ is a cubic weak bi-ideal of R.

Conversly, assume that $A = \langle \overline{\eta}, \omega \rangle$ is a cubic weak bi-ideal of R. Let $x, y, z \in H$ and $\alpha, \beta \in \Gamma$ be such that $\overline{\eta}(x) = \overline{\eta}(y) = \overline{\eta}(z) = [p_1, p_2]$ and $\omega(x) = \omega(y) = \omega(z) =$ $1 - p$. Since $\overline{\eta}$ is an interval-valued fuzzy weak bi-ideal of Γ-near-ring R, we have $\overline{\eta}(x - y) \geq \min\{\overline{\eta}(x), \overline{\eta}(y)\} = [p_1, p_2]$ and ω is a fuzzy weak bi-ideals of Γ-near-ring R, we have $\omega(x - y) \leq \max\{\omega(x), \omega(y)\} = 1 - p$.

Again, $\overline{\eta}(x\alpha y\beta z) \geq \min\{\overline{\eta}(x), \overline{\eta}(y), \overline{\eta}(z)\} = [p_1, p_2]$ and $\omega(x\alpha y\beta z) \leq \max\{\omega(x), \omega(y), \omega(z)\} = 1 - p$. So $x - y, x\alpha y\beta z \in H$.

Hence H is a weak bi-ideal of Γ-near-ring R. □

Theorem 3.11. *The direct product of cubic ideals of* Γ*-near-ring is a cubic ideal of* Γ*-near-ring.*

Proof. Let $A_i = \langle \overline{\eta}_i, \omega_i \rangle$ be cubic ideals of Γ-near-rings R_i for $i = 1, 2, 3, \ldots, n$. Let $R = R_1 \times R_2 \times \cdots \times R_n, \Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ and $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n), z = (z_1, z_2, \ldots, z_n) \in N, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \Gamma$.
$\bar{\eta}_i(x - y) = \bar{\eta}_i((x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n))$
$= \bar{\eta}_i(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$

$= \min\{\bar\eta_1(x_1-y_1), \bar\eta_2(x_2-y_2), \ldots, \bar\eta_n(x_n-y_n)\}$
$\geq \min\{\min\{\bar\eta_1(x_1), \bar\eta_1(y_1)\}, \min\{\bar\eta_2(x_2), \bar\eta_2(y_2)\}, \ldots, \min\{\bar\eta_n(x_n), \bar\eta_n(y_n)\}\}$
$= \min\{\min\{\bar\eta_1(x_1), \bar\eta_2(x_2), \ldots, \bar\eta_n(x_n)\}, \min\{\bar\eta_1(y_1), \bar\eta_2(y_2), \ldots, \bar\eta_n(y_n)\}\}$
$= \min\{(\bar\eta_1 \times \bar\eta_2 \times, \ldots, \times\bar\eta_n)(x_1, x_2, \ldots, x_n), (\bar\eta_1 \times \bar\eta_2 \times, \ldots, \times\bar\eta_n)(y_1, y_2, \ldots, y_n)\}$
$= \min\{\bar\eta_i(x), \bar\eta_i(y)\}$
$\omega_i(x-y) = \omega_i((x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n))$
$= \omega_i(x_1-y_1, x_2-y_2, \ldots, x_n-y_n)$
$= \max\{\omega_1(x_1-y_1), \omega_2(x_2-y_2), \ldots, \omega_n(x_n-y_n)\}$
$\leq \max\{\max\{\omega_1(x_1), \omega_1(y_1)\}, \max\{\omega_2(x_2), \omega_2(y_2)\}, \ldots, \max\{\omega_n(x_n), \omega_n(y_n)\}\}$
$= \max\{\max\{\omega_1(x_1), \omega_2(x_2), \ldots, \omega_n(x_n)\}, \max\{\omega_1(y_1), \omega_2(y_2), \ldots, \omega_n(y_n)\}\}$
$= \max\{(\omega_1 \times \omega_2 \times, \ldots, \times\omega_n)(x_1, x_2, \ldots, x_n), (\omega_1 \times \omega_2 \times, \cdots, \times\omega_n)(y_1, y_2, \ldots, y_n)\}$
$= \max\{\omega_i(x), \omega_i(y)\}$
$\bar\eta_i(x\alpha y\beta z)$
$= \bar\eta_i((x_1, x_2, \ldots, x_n)(\alpha_1, \alpha_2, \ldots, \alpha_n)(y_1, y_2, \ldots, y_n)(\beta_1, \beta_2, \ldots, \beta_n)(z_1, z_2, \ldots, z_n))$
$= \bar\eta_i(x_1\alpha_1y_1\beta_1z_1, x_2\alpha_2y_2\beta_2z_2, \ldots, x_n\alpha_ny_n\beta_nz_n)$
$= \min\{\bar\eta_1(x_1\alpha_1y_1\beta_1z_1), \bar\eta_2(x_2\alpha_2y_2\beta_2z_2), \ldots, \bar\eta_n(x_n\alpha_ny_n\beta_nz_1)\}$
$\geq \min\{\min\{\bar\eta_1(x_1), \bar\eta_1(y_1), \bar\eta_1(z_1)\}, \min\{\bar\eta_2(x_2), \bar\eta_2(y_2), \bar\eta_2(z_2)\}, \ldots,$
$\min\{\bar\eta_n(x_n), \bar\eta_n(y_n), \bar\eta_n(y_n)\}\}$
$= \min\{\min\{\bar\eta_1(x_1), \bar\eta_2(x_2), \ldots, \bar\eta_n(x_n)\}, \min\{\bar\eta_1(y_1), \bar\eta_2(y_2), \ldots, \bar\eta_n(y_n)\},$
$\min\{\bar\eta_1(z_1), \bar\eta_2(z_2), \ldots, \bar\eta_n(z_n)\}\}$
$= \min\{(\bar\eta_1 \times \bar\eta_2 \times, \ldots, \times\bar\eta_n)(x_1, x_2, \ldots, x_n), (\bar\eta_1 \times \bar\eta_2 \times, \ldots, \times\bar\eta_n)(y_1, y_2, \ldots, y_n),$
$(\bar\eta_1 \times \bar\eta_2 \times, \ldots, \times\bar\eta_n)(z_1, z_2, \ldots, z_n)\}$
$= \min\{\bar\eta_i(x), \bar\eta_i(y), \bar\eta_i(z)\}.$
$\omega_i(x\alpha y\beta z)$
$= \omega_i((x_1, x_2, \ldots, x_n)(\alpha_1, \alpha_2, \ldots, \alpha_n)(y_1, y_2, \ldots, y_n)(\beta_1, \beta_2, \ldots, \beta_n)(z_1, z_2, \ldots, z_n))$
$= \omega_i(x_1\alpha_1y_1\beta_1z_1, x_2\alpha_2y_2\beta_2z_2, \ldots, x_n\alpha_ny_n\beta_nz_n)$
$= \max\{\omega_1(x_1\alpha_1y_1\beta_1z_1), \omega_2(x_2\alpha_2y_2\beta_2z_2), \ldots, \omega_n(x_n\alpha_ny_n\beta_nz_1)\}$
$\leq \max\{\max\{\omega_1(x_1), \omega_1(y_1), \omega_1(z_1)\}, \max\{\omega_2(x_2), \omega_2(y_2), \omega_2(z_2)\}, \ldots,$
$\max\{\omega_n(x_n), \omega_n(y_n), \omega_n(y_n)\}\}$
$= \max\{\max\{\omega_1(x_1), \omega_2(x_2), \ldots, \omega_n(x_n)\}, \max\{\omega_1(y_1), \omega_2(y_2), \ldots, \omega_n(y_n)\},$
$\max\{\omega_1(z_1), \omega_2(z_2), \ldots, \omega_n(z_n)\}\}$
$= \max\{(\omega_1 \times \omega_2 \times, \ldots, \times\omega_n)(x_1, x_2, \ldots, x_n), (\omega_1 \times \omega_2 \times, \ldots, \times\omega_n)(y_1, y_2, \ldots, y_n),$
$(\omega_1 \times \omega_2 \times, \ldots, \times\omega_n)(z_1, z_2, \ldots, z_n)\}$
$= \max\{\omega_i(x), \omega_i(y), \omega_i(z)\}.$ □

4. Homomorphism of cubic weak bi-ideals of Γ-near-rings

Definition 4.1. [5] Let R and S be near-rings. A map $\theta : R \to S$ is called a (near-ring) homomorphism if $\theta(x+y) = \theta(x) + \theta(y)$ and $\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in R$.

Definition 4.2. [18] Let R and S be Γ-near-rings. A map $\theta : R \to S$ is called a (Γ-near-ring) homomorphism if $\theta(x+y) = \theta(x) + \theta(y)$ and $\theta(x\alpha y) = \theta(x)\ \alpha\ \theta(y)$ for all $x, y \in R$ and $\alpha \in \Gamma$.

Theorem 4.3. *Let $f : R \to R_1$ be a homomorphism between two Γ-near-rings R and R_1. If $A = \langle\bar\eta, \omega\rangle$ is a cubic weak bi-ideal of Γ-near-ring R_1, then $C_f{}^{-1}(A) = \langle C_f{}^{-1}(\bar\eta), C_f{}^{-1}(\omega)\rangle$ is a cubic weak bi-ideal of Γ-near-ring R.*

Proof. Let $A = \langle\bar\eta, \omega\rangle$ be a cubic weak bi-ideal of Γ-near-ring R_1.

Let $x, y, z \in R$. Then $C_f(x), C_f(y), C_f(z) \in R_1$, we have $\overline{\eta}$ is an interval-valued fuzzy weak bi-ideal of Γ-near-ring R_1.

$$\begin{aligned} C_f^{-1}(\overline{\eta})(x-y) &= \overline{\eta}(f(x-y)) \\ &= \overline{\eta}(f(x)-f(y)) \\ &\geq \min\{\overline{\eta}(f(x)), \overline{\eta}(f(y))\} \\ &= \min\{C_f^{-1}(\overline{\eta})(x), C_f^{-1}(\overline{\eta})(y)\} \end{aligned}$$

and ω is a fuzzy weak bi-ideal of Γ-near-ring R_1.

$$\begin{aligned} C_f^{-1}(\omega)(x-y) &= \omega(f(x-y)) \\ &= \omega(f(x)-f(y)) \\ &\leq \max\{\omega(f(x)), \omega(f(y))\} \\ &= \max\{C_f^{-1}(\omega)(x), C_f^{-1}(\omega)(y)\} \end{aligned}$$

${C_f}^{-1}(A) = \left\langle {C_f}^{-1}(\overline{\eta}), {C_f}^{-1}(\omega)\right\rangle$ is a cubic subgroup of Γ-near-ring R. Again,

$$\begin{aligned} C_f^{-1}(\overline{\eta})(x\alpha y\beta z) &= \overline{\eta}(f(x\alpha y\beta z)) \\ &= \overline{\eta}(f(x)\alpha f(y)\beta f(z)) \\ &\geq \min\{\overline{\eta}(f(x)), \overline{\eta}(f(y)), \overline{\eta}(f(z))\} \\ &= \min\{C_f^{-1}(\overline{\eta})(x), C_f^{-1}(\overline{\eta})(y), C_f^{-1}(\overline{\eta})(z)\} \\ C_f^{-1}(\omega)(x\alpha y\beta z) &= \omega(f(x\alpha y\beta z)) \\ &= \omega(f(x)\alpha f(y)\beta f(z)) \\ &\leq \max\{\omega(f(x)), \omega(f(y)), \omega(f(z))\} \\ &= \max\{C_f^{-1}(\omega)(x), C_f^{-1}(\omega)(y), C_f^{-1}(\omega)(z)\} \end{aligned}$$

Hence, ${C_f}^{-1}(A) = \left\langle {C_f}^{-1}(\overline{\eta}), {C_f}^{-1}(\omega)\right\rangle$ is a cubic weak bi-ideal of Γ-near-ring R. □

Remark 4.4. We can also state the converse of the theorem by strengthening the condition of f as follows.

Theorem 4.5. *Let $f : R \to R_1$ be a homomorphism between two Γ-near-rings R and R_1. Let $A = \langle\overline{\eta}, \omega\rangle$ is a cubic subset of Γ-near-ring R_1.*
If ${C_f}^{-1}(A) = \left\langle {C_f}^{-1}(\overline{\eta}), {C_f}^{-1}(\omega)\right\rangle$ is a cubic weak bi-ideal of Γ-near-ring R, then $A = \langle\overline{\eta}, \omega\rangle$ is a cubic weak bi-ideal of Γ-near-ring R_1.

Proof. Let $A = \langle \overline{\eta}, \omega \rangle$ be a cubic subset of Γ-near-ring R_1 and $x, y, z \in R_1$. Then $f(a) = x, f(b) = y, f(c) = z$ for some $a, b, c \in R$, it follows that $\overline{\eta}$ is an interval-valued fuzzy weak bi-ideal of Γ-near-ring R_1

$$\begin{aligned}
\overline{\eta}(x-y) &= \overline{\eta}(f(a)-f(b)) \\
&= \overline{\eta}(f(a-b)) \\
&= (C_f{}^{-1}(\overline{\eta}))(a-b) \\
&\geq \min\{C_f{}^{-1}(\overline{\eta})(a), C_f{}^{-1}(\overline{\eta})(b)\} \\
&= \min\{(\overline{\eta})(f(a)), (\overline{\eta})(f(b))\} \\
&= \min\{\overline{\eta}(x), \overline{\eta}(y)\} \\
\omega(x-y) &= \omega(f(a)-f(b)) \\
&= \omega(f(a-b)) \\
&= (C_f{}^{-1}(\omega))(a-b) \\
&\leq \max\{C_f{}^{-1}(\omega)(a), C_f{}^{-1}(\omega)(b)\} \\
&= \max\{(\omega)(f(a)), (\omega)(f(b))\} \\
&= \max\{\omega(x), \omega(f(y))\} \\
\overline{\eta}(x\alpha y\beta z) &= \overline{\eta}(f(a)\alpha f(b)\beta f(c)) \\
&= \overline{\eta}(f(a\alpha b\beta c)) \\
&= (C_f{}^{-1}(\overline{\eta}))(a\alpha b\beta c) \\
&\geq \min\{C_f{}^{-1}(\overline{\eta})(a), C_f{}^{-1}(\overline{\eta})(b), C_f{}^{-1}(\overline{\eta})(c)\} \\
&= \min\{\overline{\eta}(f(a)), \overline{\eta}(f(b)), \overline{\eta}(f(c))\} \\
&= \min\{\overline{\eta}(x), \overline{\eta}(y), \overline{\eta}(z)\} \\
\omega(x\alpha y\beta z) &= \omega(f(a)\alpha f(b)\beta f(c)) \\
&= \omega(f(a\alpha b\beta c)) \\
&= (C_f{}^{-1}(\omega))(a\alpha b\beta c) \\
&\leq \max\{C_f{}^{-1}(\omega)(a), C_f{}^{-1}(\omega)(b), C_f{}^{-1}(\omega)(c)\} \\
&= \max\{\omega(f(a)), \omega(f(b)), \omega(f(c))\} \\
&= \max\{\omega(x), \omega(y), \omega(z)\}
\end{aligned}$$

Hence, $A = \langle \overline{\eta}, \omega \rangle$ is a cubic weak bi-ideal of Γ-near-ring R_1. □

Theorem 4.6. *Let $f : R \to R_1$ be an onto Γ-near-ring homomorphism. If $A = \langle \overline{\eta}, \omega \rangle$ is a cubic weak bi-ideal of Γ-near-ring R then $C_f(A) = \langle C_f(\overline{\eta}), C_f(\omega) \rangle$ is a cubic weak bi-ideal of Γ-near-ring R_1.*

Proof. Let $A = \langle \overline{\eta}, \omega \rangle$ is a cubic weak bi-ideal of Γ-near-ring R.
Since $C_f(\overline{\eta})(x') = \sup_{f(x)=x'} (\overline{\eta}(x))$ for $x\prime \in R_1$ and $C_f(\omega)(x') = \inf_{f(x)=x'} (\omega(x))$ for $x' \in R_1$.

So, $C_f(A) = \langle C_f(\overline{\eta}), C_f(\omega)\rangle$ is non-empty. Let $x', y', z' \in R_1$. Then we have

$$\begin{aligned}
C_f(\overline{\eta})(x'-y') &= \sup_{f(p)=x'-y'} \overline{\eta}(p) \\
&\geq \sup_{f(x)=x', f(y)=y'} \overline{\eta}(x-y) \\
&\geq \sup_{f(x)=x', f(y)=y'} \min\{\overline{\eta}(x), \overline{\eta}(y)\} \\
&= \min\left\{\sup_{f(x)=x'} \overline{\eta}(x), \sup_{f(y)=y'} \overline{\eta}(y)\right\} \\
&= \min\{C_f(\overline{\eta})(x'), C_f(\overline{\eta})(y')\} \\
C_f(\omega)(x'-y') &= \inf_{f(p)=x'-y'} \omega(p) \\
&\leq \inf_{f(x)=x', f(y)=y'} \omega(x-y) \\
&\leq \inf_{f(x)=x', f(y)=y'} \max\{\omega(x), \omega(y)\} \\
&= \max\left\{\inf_{f(x)=x'} \omega(x), \inf_{f(y)=y'} \omega(y)\right\} \\
&= \max\{C_f(\omega)(x'), C_f(\omega)(y')\}
\end{aligned}$$

$$\begin{aligned}
C_f(\overline{\eta})(x'\alpha y'\beta z') &= \sup_{f(p)=x'\alpha y'\beta z'} \overline{\eta}(p) \\
&\geq \sup_{f(x)=x', f(y)=y', f(z)=z'} \overline{\eta}(x\alpha y\beta z) \\
&\geq \sup_{f(x)=x', f(y)=y', f(z)=z'} \min\{\overline{\eta}(x), \overline{\eta}(y), \overline{\eta}(z)\} \\
&= \min\left\{\sup_{f(x)=x'} \overline{\eta}(x), \sup_{f(y)=y'} \overline{\eta}(y), \sup_{f(z)=z'} \overline{\eta}(z)\right\} \\
&= \min\{C_f(\overline{\eta})(x'), C_f(\overline{\eta})(y'), C_f(\overline{\eta})(z')\} \\
C_f(\omega)(x'\alpha y'\beta z') &= \inf_{f(p)=x'\alpha y'\beta z'} \omega(p) \\
&\leq \inf_{f(x)=x', f(y)=y', f(z)=z'} \omega(x\alpha y\beta z) \\
&\leq \inf_{f(x)=x', f(y)=y', f(z)=z'} \max\{\omega(x), \omega(y), \omega(z)\} \\
&= \max\left\{\inf_{f(x)=x'} \omega(x), \inf_{f(y)=y'} \omega(y), \inf_{f(z)=z'} \omega(z)\right\} \\
&= \max\{C_f(\omega)(x'), C_f(\omega)(y'), C_f(\omega)(z')\}
\end{aligned}$$

Hence, $C_f(A) = \langle C_f(\overline{\eta}), C_f(\omega)\rangle$ is a cubic weak bi-ideal of Γ-near-ring R_1. □

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