CUBIC WEAK BI-IDEALS OF Γ-NEAR-RINGS

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ABSTRACT. In this paper, we introduced the new notion of cubic weak bi-ideals of Γ-near-rings, which is the generalized concept of cubic weak bi-ideals of near-rings. We also investigated some of its properties with examples.

1. INTRODUCTION

Zadeh [21] initiated the concept of fuzzy sets in 1965. Near-ring theory was introduced by Pilz [14]. Gamma-near-ring was introduced by Satyanarayana [15] in 1984. The concept of bi-ideals was applied to near rings and Gamma-near-rings [17, 18]. Kim et al. [6] defined the concept of fuzzy R-subgroups of near-rings. The idea of fuzzy ideals of near-rings was first proposed by Kim et al. [5]. Moreover, Manikantan [7] introduced the notion of fuzzy bi-ideals of near-rings and discussed some of its properties. Yong Uk Cho et al. [20] introduced the concept of weak bi-ideals applied to near-rings. Chinnadurai et al. [4] introduced fuzzy weak bi-ideals of near-rings. Thillaigovindan et al. [19] introduced interval valued fuzzy ideals of near rings. Jun et al. [10] introduced the concept of cubic subgroups. This structure encompasses interval-valued fuzzy set and fuzzy set. Also Jun et al. [12] introduced the notion of cubic ideals of semigroups. Chinnadurai et al. [3] introduced the notion of cubic weak bi-ideals of near-rings. In this paper, we defined a new notion of cubic weak bi-ideals of Γ-near-rings, we also discussed some of its properties with examples.

2. PRELIMINARIES

In this section, we listed some basic definitions related to cubic weak bi-ideals of Γ-near-rings. Throughout this paper R denotes a left Γ-near-ring.

Definition 2.1. [1] A near-ring is an algebraic system \((R, +, \cdot)\) consisting of a non-empty set \(R\) together with two binary operations called + and \(\cdot\) such that \((R, +)\) is a group not necessarily abelian and \((R, \cdot)\) is a semigroup connected by the following distributive law: \(x \cdot (y + z) = x \cdot y + x \cdot z\) valid for all \(x, y, z \in R\). We use the word 'near-ring' to means 'left near-ring'. We denote \(xy\) instead of \(x \cdot y\). An ideal \(I\) of a near-ring \(R\) is a subset of \(R\) such that (i) \((I, +)\) is a normal subgroup of \((R, +)\) (ii) \(RI \subseteq I\) (iii) \((x + a)y - xy \in I\) for any \(a \in I\) and \(x, y \in R\). A R-subgroup \(H\) of a near-ring \(R\) is the subset of \(R\) such that (i) \((H, +)\) is a subgroup of \((R, +)\) (ii) \(RH \subseteq H\) (iii) \(HR \subseteq H\).

Note that \(H\) is a left R-subgroup of \(R\) if \(H\) satisfies (i) and (ii) and a right R-subgroup of \(R\) if \(H\) satisfies (i) and (iii).

Key words and phrases. Near-rings, Γ-near-rings, weak bi-ideals, fuzzy weak bi-ideals, cubic weak bi-ideals, homomorphisam of cubic weak bi-ideals.
Definition 2.2. [7] Let $R$ be a near-ring. Given two subsets $A$ and $B$ of $R$, we define the following products $AB = \{ab \mid a \in A, b \in B\}$ and $A \ast B = \{(a' + b)a - a'a \mid a, a' \in A, b \in B\}$.

Definition 2.3. [17] A subgroup $B$ of $(R, +)$ is said to be bi-ideal of $R$ if $BRB \cap B \ast RB \subseteq B$.

Definition 2.4. [20] A subgroup $B$ of $(R, +)$ is said to be weak bi-ideal of $R$ if $BBB \subseteq B$.

Definition 2.5. [16] Let $(M, +)$ be a group and $\Gamma$ be a non-empty set. Then $M$ is said to be $\Gamma$-near-ring, if there exists a mapping $M \times \Gamma \times M \rightarrow M$ (The image of $(x, \alpha, y)$ is denoted by $x \alpha y$) satisfying the following conditions:
1. $(x + y)\alpha z = x\alpha z + y\alpha z$,
2. $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition 2.6. [16] Let $M$ be a $\Gamma$-near-ring. A normal subgroup $(I, +)$ of $(M, +)$ is called
1. a left ideal if $x\alpha(y + i) - x\alpha y \in I$,
2. a right ideal if $iax \in I$ for all $x, y \in M, \alpha \in \Gamma, i \in I$,
3. an ideal if it is both a left ideal and a right ideal of $M$.

A $\Gamma$-near-ring $M$ is said to be zero-symmetric if $aa0 = 0$ for all $a \in M$, $\alpha \in \Gamma$, where 0 is the additive identity in $M$.

Definition 2.7. [18] A subgroup $B$ of $(M, +)$ is a bi-ideal iff $B\Gamma M\Gamma B \subseteq B$.

Definition 2.8. [15] Let $M$ be a $\Gamma$-near-ring. Given two subsets $A$ and $B$ of $M$, we define the following products $A\Gamma B = \{a\alpha b \mid a \in A, b \in B \text{ and } \alpha \in \Gamma\}$ and also define another operation on $\ast$ on the class of subset $M$ is defined by $A\Gamma \ast B = \{(a' + b)\gamma a - a'\gamma a \mid a, a' \in A, b \in B \text{ and } \gamma \in \Gamma\}$.

Definition 2.9. [2] A fuzzy subset $\mu$ of a set $X$ is a function $\mu : X \rightarrow [0, 1]$.

Definition 2.10. [2] Let $\mu$ and $\lambda$ be any two fuzzy subsets of $R$. Then $\mu\lambda$ is fuzzy subset of $R$ denoted by

$$(\mu\lambda)(x) = \begin{cases} \sup_{x = yz} \min\{\mu(y), \lambda(z)\} & \text{if } x = yz \text{ for all } x, y, z \in R \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.11. [7] A fuzzy subgroup $\mu$ of $(R, +)$ is said to be fuzzy bi-ideal of $R$ if $\mu R\mu \cap \mu \ast R\mu \subseteq \mu$.

Definition 2.12. [1] Let $R$ be a near-ring and $\mu$ be a fuzzy subset of $R$. We say $\mu$ is a fuzzy subnear-ring of $R$ if
1. $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$
2. $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in R$.

Definition 2.13. [1] Let $R$ be a near-ring and $\mu$ be a fuzzy subset of $R$. Then $\mu$ is called a fuzzy ideal of $R$, if
1. $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$
2. $\mu(y + x - y) \geq \mu(x)$
3. $\mu(xy) \geq \mu(y)$
4. $\mu((x + z)y - xy) \geq \mu(z)$ for all $x, y \in R$.

A fuzzy subset with (1) to (3) is called a fuzzy left ideal of $R$, whereas a fuzzy subset with (1),(2) and (4) are called a fuzzy right ideal of $R$. 
Definition 2.14. [1] A fuzzy subset \( \mu \) of a near-ring \( R \) is called a fuzzy \( R \)-subgroup of \( R \) if
1. \( \mu \) is a fuzzy subgroup of \((R, +)\)
2. \( \mu(xy) \geq \mu(y) \)
3. \( \mu(xy) \geq \mu(x) \) for all \( x, y \in R \).

A fuzzy subset with (1) and (2) is called a fuzzy left \( R \)-subgroup of \( R \), whereas a fuzzy subset with (1) and (3) is called a fuzzy right \( R \)-subgroup of \( R \).

Definition 2.15. [4] A fuzzy subgroup \( \mu \) of \( R \) is called fuzzy weak bi-ideal of \( R \), if

\[ \mu(xy) \geq \min \{\mu(x), \mu(y), \mu(z)\}. \]

Definition 2.16. [2] Let \( X \) be a non-empty set. A mapping \( \bar{\mu} : X \to D[0,1] \) is called an interval-valued (in short i-v) fuzzy subset of \( X \), if for all \( x \in X, \bar{\mu}(x) = \{\mu^{-}(x), \mu^{+}(x)\} \), where \( \mu^{-} \) and \( \mu^{+} \) are fuzzy subsets of \( X \) such that \( \mu^{-}(x) \leq \mu^{+}(x) \).

Thus \( \bar{\mu}(x) \) is an interval (a closed subset of \([0,1]\)) and not a number from the interval \([0,1]\) as in the case of fuzzy set.

Definition 2.17. [3] A cubic set \( A = \langle \bar{\mu}, \omega \rangle \) of \( R \) is called cubic subgroup of \( R \), if
1. \( \bar{\mu}(x - y) \geq \min \{\bar{\mu}(x), \bar{\mu}(y)\} \)
2. \( \omega(x - y) \leq \max \{\omega(x), \omega(y)\} \) \( \forall x, y \in R \).

Definition 2.18. [3] A cubic subgroup \( A = \langle \bar{\mu}, \omega \rangle \) of \( R \) is called cubic weak bi-ideal of \( R \), if
1. \( \bar{\mu}(xyz) \geq \min \{\bar{\mu}(x), \bar{\mu}(y), \bar{\mu}(z)\} \)
2. \( \omega(xyz) \leq \max \{\omega(x), \omega(y), \omega(z)\} \) \( \forall x, y, z \in R \).

Definition 2.19. [3] Let \( A_i \) be cubic weak bi-ideals of near-rings \( R_i \) for \( i = 1, 2, 3, \ldots, n \). Then the cubic direct product of \( A_i(i = 1, 2, \ldots, n) \) is a function \( \bar{\mu}_1 \times \bar{\mu}_2 \times \cdots \times \bar{\mu}_n : R_1 \times R_2 \times \cdots \times R_n \to D[0,1], \)

\( \omega_1 \times \omega_2 \times \cdots \times \omega_n : R_1 \times R_2 \times \cdots \times R_n \to [0,1] \) defined by

\( (\bar{\mu}_1 \times \bar{\mu}_2 \times \cdots \times \bar{\mu}_n)(x_1, x_2, \ldots, x_n) = \min \{\bar{\mu}_1(x_1), \bar{\mu}_2(x_2), \ldots, \bar{\mu}_n(x_n)\} \)

and

\( (\omega_1 \times \omega_2 \times \cdots \times \omega_n)(x_1, x_2, \ldots, x_n) = \max \{\omega_1(x_1), \omega_2(x_2), \ldots, \omega_n(x_n)\} \).

Definition 2.20. [3] Let \( A_1 = \langle \bar{\mu}_1, \omega_1 \rangle \) and \( A_2 = \langle \bar{\mu}_2, \omega_2 \rangle \) be any two cubic subsets of \( R \). Then \( A_1 \times A_2 \) is cubic subsets of \( R \) defined by:

\[
(A_1 \times A_2)(x) = \begin{cases}
\sup_{x=yz} \min \{\bar{\mu}_1(y), \bar{\mu}_2(z)\} & \text{if } x = yz \text{ for all } x, y, z \in R \\
[0,0] & \text{otherwise}
\end{cases}
\]

\[
(\omega_1 \times \omega_2)(x) = \begin{cases}
\inf_{x=yz} \max \{\omega_1(y), \omega_2(z)\} & \text{if } x = yz \text{ for all } x, y, z \in R \\
1 & \text{otherwise}
\end{cases}
\]

Definition 2.21. [9] Let \( f \) be a mapping from a set \( X \) to \( Y \) and \( A = \langle \bar{\eta}, \lambda \rangle \) be a cubic set of \( X \) then the image of \( X \) (i.e.,) \( C_f(A) = \langle C_f(\bar{\eta}), C_f(\lambda) \rangle \) is a cubic set of \( Y \) defined by

\[
C_f(\bar{\mu})(y) = \begin{cases}
\sup_{f(x)=y} \bar{\eta}(x) & \text{if } f^{-1}(y) \neq \emptyset \\
[0,0] & \text{otherwise}
\end{cases}
\]

\[
C_f(\lambda)(y) = \begin{cases}
\inf_{f(x)=y} \lambda(x) & \text{if } f^{-1}(y) \neq \emptyset \\
1 & \text{otherwise}
\end{cases}
\]
and let $f$ be a mapping from a set $X$ to $Y$ and $A = \langle \eta, \lambda \rangle$ is a cubic set of $Y$, then the pre image of $Y$ (i.e.,) $C_f^{-1}(A) = \langle C_f^{-1}(\eta), C_f^{-1}(\lambda) \rangle$ is a cubic set of $X$ is defined by

$$C_f^{-1}(A)(x) = \begin{cases} C_f^{-1}(\eta)(x) = \eta(f(x)) \\ C_f^{-1}(\lambda)(x) = \lambda(f(x)) \end{cases}$$

3. Cubic weak bi-ideals of $\Gamma$-near-rings

In this section, we introduced the new notion of cubic weak bi-ideals of $\Gamma$-near-rings and discuss some of its properties.

**Definition 3.1.** A cubic set $A = \langle \eta, \omega \rangle$ of $\Gamma$-near-ring $R$ is called cubic subgroup of $R$, if

(i) $\eta(x - y) \geq \min\{\eta(x), \eta(y)\}$

(ii) $\omega(x - y) \leq \max\{\omega(x), \omega(y)\}$ $\forall x, y \in R$. 

**Definition 3.2.** A cubic subgroup $A = \langle \eta, \omega \rangle$ of a $\Gamma$-near-ring $R$ is called cubic weak bi-ideal of $\Gamma$-near-ring $R$, if

(i) $\eta(x\alpha y\beta z) \geq \min\{\eta(x), \eta(y), \eta(z)\}$

(ii) $\omega(x\alpha y\beta z) \leq \max\{\omega(x), \omega(y), \omega(z)\}$ $\forall x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

**Example 3.3.** Let $R = \{a, b, c, d\}$ be a non-empty set with binary operation $+$ and $\Gamma = \{\alpha, \beta\}$ be a non-empty set of binary operations as shown in the following tables:

<table>
<thead>
<tr>
<th>$+$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$b$</td>
<td>$c$</td>
<td>$d$</td>
</tr>
<tr>
<td>$b$</td>
<td>$b$</td>
<td>$a$</td>
<td>$a$</td>
<td>$a$</td>
</tr>
<tr>
<td>$c$</td>
<td>$c$</td>
<td>$d$</td>
<td>$a$</td>
<td>$b$</td>
</tr>
<tr>
<td>$d$</td>
<td>$d$</td>
<td>$c$</td>
<td>$b$</td>
<td>$a$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$a$</td>
<td>$a$</td>
<td>$a$</td>
<td>$a$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$a$</td>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
</tr>
</tbody>
</table>

Clearly (i) $(R, +)$ is a group (ii) $x\alpha(y + z) = x\alpha y + x\alpha z$ (iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for every $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. Then $R$ is a $\Gamma$-near-ring.

Define a cubic set $A = \langle \eta, \omega \rangle$ in $R$ as follows:

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\eta(x)$</th>
<th>$\omega(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.80.9</td>
<td>0.2</td>
</tr>
<tr>
<td>b</td>
<td>0.60.7</td>
<td>0.4</td>
</tr>
<tr>
<td>c</td>
<td>0.20.3</td>
<td>0.6</td>
</tr>
<tr>
<td>d</td>
<td>0.20.3</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Hence, $A = \langle \eta, \omega \rangle$ is a cubic weak bi-ideal of $\Gamma$-near-ring $R$.

**Definition 3.4.** Let $A_1 = \langle \eta_1, \omega_1 \rangle$ and $A_2 = \langle \eta_2, \omega_2 \rangle$ be any two cubic subsets of $R$. Then $A_1 A_2$ is a cubic subsets of $R$ defined by:

\[
(A_1 A_2)(x) = \begin{cases} \sup \min\{\eta_1(y), \eta_2(z)\} & \text{for all } x, y, z \in R \text{ and } \alpha \in \Gamma \\ [0, 0] & \text{otherwise} \end{cases}
\]

\[
(\omega_1 \omega_2)(x) = \begin{cases} \inf \max\{\omega_1(y), \omega_2(z)\} & \text{for all } x, y, z \in R \text{ and } \alpha \in \Gamma \\ 1 & \text{otherwise} \end{cases}
\]
Theorem 3.5. Let $A = \langle \eta, \omega \rangle$ be a cubic subgroup of $\Gamma$-near-ring $R$. Then $A = \langle \eta, \omega \rangle$ is a cubic weak bi-ideal of $\Gamma$-near-ring $R$ if and only if $AAA \subseteq A$.

Proof. Assume that $A = \langle \eta, \omega \rangle$ is a cubic weak bi-ideal of $\Gamma$-near-ring $R$. Let $x, y, z, p, q \in R$ and $\alpha, \beta \in \Gamma$, such that $x = yaz$ and $y = p\beta q$. Then

$$(\eta \eta \eta)(x) = \sup_{x = yaz} \{ \min \{ (\eta \eta \eta)(y), \eta(z) \} \}$$
$$= \sup_{x = yaz} \left\{ \min \left\{ \sup_{y = p\beta q} \{ \min \{ \eta(p), \eta(q) \}, \eta(z) \} \right\} \right\}$$
$$= \sup_{x = p\beta qaz} \sup_{y = p\beta q} \{ \min \{ \eta(p) \eta(q), \eta(z) \} \}$$
$$\leq \sup_{x = p\beta qaz} \eta(p\beta qaz)$$
$$= \eta(x)$$

If $x$ can not be expressed as $x = yaz$ then $(\eta \eta \eta)(x) = 0 \leq \eta(x)$. In both cases $\eta \eta \eta \subseteq \eta$.

$$(\omega \omega \omega)(x) = \inf_{x = yaz} \{ \max \{ (\omega \omega \omega)(y), \omega(z) \} \}$$
$$= \inf_{x = yaz} \left\{ \max \left\{ \inf_{y = p\beta q} \{ \max \{ \omega(p), \omega(q) \}, \omega(z) \} \right\} \right\}$$
$$= \inf_{x = p\beta qaz} \inf_{y = p\beta q} \{ \max \{ \omega(p), \omega(q), \omega(z) \} \}$$
$$\geq \inf_{x = p\beta qaz} \omega(p\beta qaz)$$
$$= \omega(x)$$

If $x$ can not be expressed as $x = yaz$ then $(\omega \omega \omega)(x) = 1 \geq \omega(x)$. In both cases $\omega \omega \omega \cup \omega$.

Hence $AAA \subseteq A$.

Conversely, assume that $AAA \subseteq A$ holds. To prove that $A = \langle \eta, \omega \rangle$ is a cubic weak bi-ideal of $\Gamma$-near-ring $R$.

For any $x, y, z, a \in R$ and $\alpha, \alpha_1, \beta_1 \in \Gamma$, such that $a = x\alpha y\beta z$ then
$$\eta(x\alpha y\beta z) = \eta(a) \geq (\eta \eta \eta)(a)$$
$$= \sup_{a = b_0 c} \min \{ (\eta \eta \eta)(b), \eta(c) \}$$
$$= \sup_{a = b_0 c} \left\{ \min \left\{ \sup_{b = p\beta_1 q} \{ \min \{ \eta(p), \eta(q) \}, \eta(c) \} \right\} \right\}$$
$$= \sup_{a = p\beta_1 q \alpha_1 c} \{ \min \{ \eta(p), \eta(q), \eta(c) \} \}$$
$$\eta(x\alpha y\beta z) \geq \min \{ \eta(x), \eta(y), \eta(z) \}$$
$$\omega(x\alpha y\beta z) = \omega(a) \leq (\omega \omega \omega)(a)$$
$$= \inf_{a = b_0 c} \max \{ (\omega \omega)(b), \omega(c) \}$$
It follows that $(\omega \eta)$.

**Proof.** Let $A_1 = (\bar{\eta}_1, \omega)$ and $A_2 = (\bar{\eta}_2, \omega_2)$ be two cubic weak bi-ideals of $\Gamma$-near-ring $R$.

Since $\bar{\eta}_1$ and $\bar{\eta}_2$ are interval-valued fuzzy weak bi-ideals of $\Gamma$-near-ring $R$ then
\[
(\bar{\eta}_1 \bar{\eta}_2)(x - y) = \sup_{x - y = p_1 \alpha_1 q_1 - p_2 \alpha_2 q_2} \min \{\bar{\eta}_1(p), \bar{\eta}_2(q)\}
\geq \sup_{x - y = p_1 \alpha_1 q_1 - p_2 \alpha_2 q_2} \min \{\bar{\eta}_1(p_1), \bar{\eta}_1(p_2)\} \min \{\bar{\eta}_2(q_1), \bar{\eta}_2(q_2)\}
= \sup \min \{\bar{\eta}_1(p_1), \bar{\eta}_2(q_1)\} \min \{\bar{\eta}_1(p_2), \bar{\eta}_2(q_2)\}
= \min \left\{ \sup \min \{\bar{\eta}_1(p_1), \bar{\eta}_2(q_1)\}, \sup \min \{\bar{\eta}_1(p_2), \bar{\eta}_2(q_2)\} \right\}
= \min \{\bar{\eta}_1 \bar{\eta}_2(x), (\bar{\eta}_1 \bar{\eta}_2)(y)\}
\]

It follows that $(\bar{\eta}_1 \bar{\eta}_2)$ is an interval-valued fuzzy subgroup of $\Gamma$-near-ring $R$. Further
\[
(\bar{\eta}_1 \bar{\eta}_2)(\bar{\eta}_1 \bar{\eta}_2) = \bar{\eta}_1 \bar{\eta}_2(\bar{\eta}_1 \bar{\eta}_2) \bar{\eta}_2
\subseteq \bar{\eta}_1 \bar{\eta}_2 \bar{\eta}_2 \bar{\eta}_2 \bar{\eta}_2
\subseteq \bar{\eta}_1 \bar{\eta}_2
\subseteq (\bar{\eta}_1 \bar{\eta}_2)
\]

Therefore $(\bar{\eta}_1 \bar{\eta}_2)$ is an interval-valued fuzzy weak bi-ideals of $\Gamma$-near-ring $R$.

Since $\omega_1, \omega_2$ are fuzzy weak bi-ideals of $\Gamma$-near-ring $R$, then
\[
(\omega_1 \omega_2)(x - y) = \inf_{x - y = p_1 \alpha_1 q_1 - p_2 \alpha_2 q_2} \max \{\omega_1(p), \omega_2(q)\}
\leq \inf_{x - y = p_1 \alpha_1 q_1 - p_2 \alpha_2 q_2} \max \{\omega_1(p_1), \omega_2(q_1)\} \max \{\omega_1(p_2), \omega_2(q_2)\}
= \inf \max \{\omega_1(p_1), \omega_2(q_1)\} \max \{\omega_1(p_2), \omega_2(q_2)\}
= \max \left\{ \inf \max \{\omega_1(p_1), \omega_2(q_1)\}, \inf \max \{\omega_1(p_2), \omega_2(q_2)\} \right\}
= \max \{\omega_1 \omega_2(x), (\omega_1 \omega_2)(y)\}
\]

It follows that $(\omega_1 \omega_2)$ is a fuzzy subgroup of $\Gamma$-near-ring $R$. Further
\[
(\omega_1 \omega_2)(\omega_1 \omega_2) = \omega_1 \omega_2(\omega_1 \omega_2 \omega_1) \omega_2
\supseteq \omega_1 \omega_2(\omega_1 \omega_2 \omega_1) \omega_2
\supseteq \omega_1 (\omega_1 \omega_2) \omega_2
\supseteq (\omega_1 \omega_2)
\]
Thus, $(\omega_1, \omega_2)$ is a fuzzy weak bi-ideal of $\Gamma$-near-ring $R$.

Hence, $A_1A_2 = (\langle \eta_1, \eta_2 \rangle, (\omega_1, \omega_2))$ is a cubic weak bi-ideal of $\Gamma$-near-ring $R$. \qed

**Remark 3.7.** Let $A_1$ and $A_2$ be two cubic weak bi-ideals of $\Gamma$-near-ring $R$ then the product $A_2A_1$ is also a cubic weak bi-ideal of $\Gamma$-near-ring $R$.

**Theorem 3.8.** Let $A = \langle \eta_1, \omega_2 \rangle$ be a cubic weak bi-ideal of $\Gamma$-near-ring $R$, then the set $R_A = \{ x \in R \mid A(x) = A(0) \}$ (i.e., $R_A = \{ x \in R \mid \eta(x) = \eta(0) \text{ and } \omega(x) = \omega(0) \}$) is a weak bi-ideal of $\Gamma$-near-ring $R$.

**Proof.** Let $A = \langle \eta, \omega \rangle$ be a cubic weak bi-ideal of $R$. Let $x, y \in R_A$.

Then $A(x) = A(0)$ and $A(y) = A(0)$, (i.e., $\eta(x) = \eta(0)$, $\omega(x) = \omega(0)$ and $\eta(y) = \eta(0)$, $\omega(y) = \omega(0)$). Since $\eta$ is an interval-valued fuzzy weak bi-ideal of $\Gamma$-near-ring $R$. We have $\eta(x) = \eta(0)$ and $\eta(y) = \eta(0)$ then $\eta(x - y) \geq \min \{ \eta(x), \eta(y) \} = \eta(0)$ and $\omega$ is a fuzzy weak bi-ideal of $\Gamma$-near-ring $R$, we have $\omega(x) = \omega(0)$ and $\omega(y) = \omega(0)$ then $\omega(x - y) \leq \max \{ \omega(x), \omega(y) \} = \omega(0)$. Thus $x - y \in R_A$ for every $x, y, z \in R_A$ and $\alpha, \beta \in \Gamma$. Then $A(x) = A(0)$, $A(y) = A(0)$ and $A(z) = A(0)$. Since $\eta$ is an interval-valued fuzzy weak bi-ideal of $\Gamma$-near-ring $R$, we have $\eta(x) = \eta(0)$, $\eta(y) = \eta(0)$ and $\eta(z) = \eta(0)$ then $\eta(x\alpha y\beta z) \geq \min \{ \eta(x), \eta(y), \eta(z) \} = \eta(0)$ and $\omega$ is a fuzzy weak bi-ideal of $\Gamma$-near-ring $R$, we have $\omega(x) = \omega(0)$, $\omega(y) = \omega(0)$, $\omega(z) = \omega(0)$ and $\omega(x\alpha y\beta z) \leq \max \{ \omega(x), \omega(y), \omega(z) \} = \max \{ \omega(0), \omega(0), \omega(0) \}$ $= \omega(0)$. Thus $x\alpha y\beta z \in R_A$. Hence $R_A$ is a weak bi-ideal of $\Gamma$-near-ring $R$. \qed

**Theorem 3.9.** Let $\{ A_i \}_{i \in \Omega} = \{ \eta_i, \omega_i : i \in \Omega \}$ be a family of cubic weak bi-ideals $\Gamma$-near-ring $R$, then $\bigcap_{i \in \Omega} A_i = \bigg\langle \bigcap_{i \in \Omega} \eta_i, \bigcup_{i \in \Omega} \omega_i \bigg\rangle$ is also a family of cubic weak bi-ideal $\Gamma$-near-ring $R$, where $\Omega$ is any index set.

**Proof.** Let $\{ A_i \}_{i \in \Omega} = \{ \eta_i, \omega_i : i \in \Omega \}$ be a family of cubic weak bi-ideals of $\Gamma$-near-ring $R$.

Let $x, y, z \in R$, $\alpha, \beta \in \Gamma$ and $\bigcap_{i \in \Omega} \eta_i(x) = (\inf_{i \in \Omega} \eta_i)(x) = \inf_{i \in \Omega} \eta_i(x)$,

$\bigcup_{i \in \Omega} \omega_i(x) = (\sup_{i \in \Omega} \omega_i)(x) = \sup_{i \in \Omega} \omega_i(x)$

Since $\eta_i$ is a family of interval-valued fuzzy weak bi-ideals of $\Gamma$-near-ring $R$, we have

$$\bigcap_{i \in \Omega} \eta_i(x - y) = \inf_{i \in \Omega} \eta_i(x - y) \geq \inf_{i \in \Omega} \min \{ \eta_i(x), \eta_i(y) \} = \min \left\{ \inf_{i \in \Omega} \eta_i(x), \inf_{i \in \Omega} \eta_i(y) \right\} = \min \left\{ \bigcap_{i \in \Omega} \eta_i(x), \bigcup_{i \in \Omega} \eta_i(y) \right\}$$
and $\omega_i$ is a family of fuzzy weak bi-ideals of $\Gamma$-near-ring $R$. We have
\[
\bigcup_{i \in \Omega} \omega_i(x - y) = \sup_{i \in \Omega} \omega_i(x - y) \\
\leq \sup_{i \in \Omega} \max\{\omega_i(x), \omega_i(y)\} \\
= \max\left\{\sup_{i \in \Omega} \omega_i(x), \sup_{i \in \Omega} \omega_i(y)\right\} \\
= \max\left\{\bigcup_{i \in \Omega} \omega_i(x), \bigcup_{i \in \Omega} \omega_i(y)\right\}
\]
Thus, $\bigcap_{i \in \Omega} A_i$ is a cubic subgroup of $\Gamma$-near-ring $R$.

Again,
\[
\bigcap_{i \in \Omega} \eta_i(x\alpha y\beta z) = \inf_{i \in \Omega} \eta_i(x\alpha y\beta z) \\
\geq \inf_{i \in \Omega} \min\{\eta_i(x), \eta_i(y), \eta_i(z)\} \\
= \min\left\{\inf_{i \in \Omega} \eta_i(x), \inf_{i \in \Omega} \eta_i(y), \inf_{i \in \Omega} \eta_i(z)\right\} \\
= \min\left\{\bigcap_{i \in \Omega} \eta_i(x), \bigcap_{i \in \Omega} \eta_i(y), \bigcap_{i \in \Omega} \eta_i(z)\right\}
\]
\[
\bigcup_{i \in \Omega} \omega_i(x\alpha y\beta z) = \sup_{i \in \Omega} \omega_i(x\alpha y\beta z) \\
\leq \sup_{i \in \Omega} \max\{\omega_i(x), \omega_i(y), \omega_i(z)\} \\
= \max\left\{\sup_{i \in \Omega} \omega_i(x), \sup_{i \in \Omega} \omega_i(y), \sup_{i \in \Omega} \omega_i(z)\right\} \\
= \max\left\{\bigcup_{i \in \Omega} \omega_i(x), \bigcup_{i \in \Omega} \omega_i(y), \bigcup_{i \in \Omega} \omega_i(z)\right\}
\]
Hence, $\bigcap_{i \in \Omega} A_i = \bigcap_{i \in \Omega} \eta_i \bigcup_{i \in \Omega} \omega_i$ is a family of cubic weak bi-ideal of $\Gamma$-near-ring $R$. \qed

**Theorem 3.10.** Let $H$ be a non empty subset of $\Gamma$-near-ring $R$ and $A = \langle \eta, \omega \rangle$ be a cubic subset of $\Gamma$-near-ring $R$ defined by
\[
A(x) = \begin{cases}
\bar{\eta}(x) = \begin{cases} [p_1, p_2] & \text{if } x \in H \\
[p_1, q_2] & \text{otherwise}
\end{cases} \\
\omega(x) = \begin{cases} 1 - p & \text{if } x \in H \\
1 - q & \text{otherwise}
\end{cases}
\end{cases}
\]
for all $x \in R, [p_1, p_2], [q_1, q_2] \in D[0, 1]$ and $p, q \in [0, 1]$ with $[p_1, p_2] \geq [q_1, q_2], p > q$. Then $H$ is a weak bi-ideal of $\Gamma$-near-ring $R$ $\iff A = \langle \eta, \omega \rangle$ is a cubic weak bi-ideal of $\Gamma$-near-ring $R$.

**Proof.** Assume that $H$ is a weak bi-ideal of $\Gamma$-near-ring $R$. Let $x, y \in H$ we consider four cases:
(1) $x \in H$ and $y \in H$
(2) $x \in H$ and $y \notin H$
(3) $x \notin H$ and $y \in H$
(4) $x \notin H$ and $y \notin H$

Case (i) If $x \in H$ and $y \in H$. Then $\eta(x) = \eta(y) = \omega(x) = 1 - p = \omega(y)$. Since $H$ is a weak bi-ideal $\Gamma$-near-ring $R$, then $x - y \in R$. Thus
\[
\eta(x - y) = \eta(y) = \omega(x - y) = 1 - p = \max\{1 - p, 1 - p\} = \max\{\omega(x), \omega(y)\}.
\]

Case (ii) If $x \in H$ and $y \notin H$. Then $\eta(x) = \eta(y) = [q_1, q_2]$ and $\omega(x) = 1 - p, \omega(y) = 1 - q$. Clearly, $\eta(x - y) = \min\{\eta(x), \eta(y)\} = \min\{[p_1, p_2], [q_1, q_2]\} = [q_1, q_2]$ and $\omega(x - y) = \max\{\omega(x), \omega(y)\} = \max\{1 - p, 1 - q\} = 1 - q$.

Now, $\eta(x - y) = [p_1, p_2]$ or $[q_1, q_2]$ according as $x - y \in H$ or $x - y \notin H$. By assumption that $[p_1, p_2] > [q_1, q_2]$ and $p > q$, we have $\eta(x - y) = \min\{\eta(x), \eta(y)\}$ and $\omega(x - y) \leq \max\{\omega(x), \omega(y)\}$.

Similarly, we can prove that case (iii).

Case (iv) If $x \notin H$ and $y \notin H$. Then $\eta(x) = [q_1, q_2] = \eta(y)$ and $\omega(x) = 1 - q = \omega(y)$. So, $\min\{\eta(x), \eta(y)\} = [q_1, q_2]$ and $\max\{\omega(x), \omega(y)\} = 1 - q$.

Next, $\eta(x - y) = [p_1, P_2]$ or $[q_1, q_2]$ and $\omega(x - y) = 1 - p$ or $1 - q$, according as $x - y \in H$ or $x - y \notin H$. So, $A = \langle \eta, \omega \rangle$ is a cubic subgroup of $R$. Now, let $x, y, z \in H$. We have the following eight cases:

1. $x \in H, y \in H$ and $z \in H$
2. $x \in H, y \notin H$ and $z \in H$
3. $x \in H, y \notin H$ and $z \notin H$
4. $x \notin H, y \in H$ and $z \notin H$
5. $x \notin H, y \notin H$ and $z \in H$
6. $x \in H, y \notin H$ and $z \notin H$
7. $x \in H, y \notin H$ and $z \notin H$
8. $x \notin H, y \notin H$ and $z \notin H$

These cases can be proved by similar arguments of the cubic cases above.

Hence, $\eta(x \circ y \circ z) \geq \min\{\eta(x), \eta(y), \eta(z)\}$ and $\omega(x \circ y \circ z) \leq \max\{\omega(x), \omega(y), \omega(z)\}$. Therefore, $A = \langle \eta, \omega \rangle$ is a cubic weak bi-ideal of $R$.

Conversely, assume that $A = \langle \eta, \omega \rangle$ is a cubic weak bi-ideal of $R$. Let $x, y, z \in H$ and $\alpha, \beta \in \Gamma$ be such that $\eta(x) = \eta(y) = \eta(z) = [p_1, p_2]$ and $\omega(x) = \omega(y) = \omega(z) = 1 - p$. Since $\eta$ is an interval-valued fuzzy weak bi-ideal of $\Gamma$-near-ring $R$, we have $\eta(x - y) \geq \min\{\eta(x), \eta(y)\} = [p_1, p_2]$ and $\omega$ is a fuzzy weak bi-ideals of $\Gamma$-near-ring $R$, we have $\omega(x - y) \leq \max\{\omega(x), \omega(y)\} = 1 - p$.

Again, $\eta(x \circ y \circ z) \geq \min\{\eta(x), \eta(y), \eta(z)\} = [p_1, p_2]$ and $\omega(x \circ y \circ z) \leq \max\{\omega(x), \omega(y), \omega(z)\} = 1 - p$. So $x - y, x \circ y \circ z \in H$.

Hence $H$ is a weak bi-ideal of $\Gamma$-near-ring $R$. □

**Theorem 3.11.** The direct product of cubic ideals of $\Gamma$-near-ring is a cubic ideal of $\Gamma$-near-ring.

**Proof.** Let $A_i = \langle \eta_i, \omega_i \rangle$ be cubic ideals of $\Gamma$-near-rings $R_i$ for $i = 1, 2, 3, \ldots, n$. Let $R = R_1 \times R_2 \times \cdots \times R_n, \Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ and $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n), z = (z_1, z_2, \ldots, z_n) \in N, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \Gamma$.

\[
\eta_i(x - y) = \eta_i((x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n))
= \eta_i(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)
\]
\[ \omega_i(x - y) = \omega_i(x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n) \]
\[ = \max\{\omega_1(x_1 - y_1), \omega_2(x_2 - y_2), \ldots, \omega_n(x_n - y_n)\} \]
\[ \leq \max\{\max\{\omega_1(x_1), \omega_1(y_1)\}, \max\{\omega_2(x_2), \omega_2(y_2)\}, \ldots, \max\{\omega_n(x_n), \omega_n(y_n)\}\} \]
\[ = \max\{\max\{\omega_1(x_1), \omega_2(x_2), \ldots, \omega_n(x_n)\}, \max\{\omega_1(y_1), \omega_2(y_2), \ldots, \omega_n(y_n)\}\} \]
\[ = \max\{\omega_1(x_1 \times \omega_2 \times \ldots \times \omega_n(x_n), \omega_1(y_1 \times \omega_2 \times \ldots \times \omega_n(y_n))\} \]
\[
\bar{\eta}_i(x \circ \bar{y} \circ \bar{z}) = \bar{\eta}_i((x_1, x_2, \ldots, x_n)(/\alpha_1, \alpha_2, \ldots, \alpha_n)(y_1, y_2, \ldots, y_n)(/\beta_1, \beta_2, \ldots, \beta_n)(z_1, z_2, \ldots, z_n))
\]
Let $x, y, z \in R$. Then $C_f(x), C_f(y), C_f(z) \in R_1$, we have $\eta$ is an interval-valued fuzzy weak bi-ideal of $\Gamma$-near-ring $R_1$.

$$C_f^{-1}(\eta)(x - y) = \eta(f(x - y))$$
$$= \eta(f(x) - f(y))$$
$$\geq \min\{\eta(f(x)), \eta(f(y))\}$$
$$= \min\{C_f^{-1}(\eta)(x), C_f^{-1}(\eta)(y)\}$$

and $\omega$ is a fuzzy weak bi-ideal of $\Gamma$-near-ring $R_1$.

$$C_f^{-1}(\omega)(x - y) = \omega(f(x - y))$$
$$= \omega(f(x) - f(y))$$
$$\leq \max\{\omega(f(x)), \omega(f(y))\}$$
$$= \max\{C_f^{-1}(\omega)(x), C_f^{-1}(\omega)(y)\}$$

$$C_f^{-1}(A) = \langle C_f^{-1}(\eta), C_f^{-1}(\omega) \rangle$$ is a cubic subgroup of $\Gamma$-near-ring $R$. Again,

$$C_f^{-1}(\eta)(x\alpha y\beta z) = \eta(f(x\alpha y\beta z))$$
$$= \eta(f(x)\alpha f(y)\beta f(z))$$
$$\geq \min\{\eta(f(x)), \eta(f(y)), \eta(f(z))\}$$
$$= \min\{C_f^{-1}(\eta)(x), C_f^{-1}(\eta)(y), C_f^{-1}(\eta)(z)\}$$

$$C_f^{-1}(\omega)(x\alpha y\beta z) = \omega(f(x\alpha y\beta z))$$
$$= \omega(f(x)\alpha f(y)\beta f(z))$$
$$\leq \max\{\omega(f(x)), \omega(f(y)), \omega(f(z))\}$$
$$= \max\{C_f^{-1}(\omega)(x), C_f^{-1}(\omega)(y), C_f^{-1}(\omega)(z)\}$$

Hence, $C_f^{-1}(A) = \langle C_f^{-1}(\eta), C_f^{-1}(\omega) \rangle$ is a cubic weak bi-ideal of $\Gamma$-near-ring $R$. $\square$

**Remark 4.4.** We can also state the converse of the theorem by strengthening the condition of $f$ as follows.

**Theorem 4.5.** Let $f : R \to R_1$ be a homomorphism between two $\Gamma$-near-rings $R$ and $R_1$. Let $A = \langle \eta, \omega \rangle$ is a cubic subset of $\Gamma$-near-ring $R_1$. If $C_f^{-1}(A) = \langle C_f^{-1}(\eta), C_f^{-1}(\omega) \rangle$ is a cubic weak bi-ideal of $\Gamma$-near-ring $R$, then $A = \langle \eta, \omega \rangle$ is a cubic weak bi-ideal of $\Gamma$-near-ring $R_1$. 


Proof. Let $A = \langle \eta, \omega \rangle$ be a cubic subset of $\Gamma$-near-ring $R_1$ and $x, y, z \in R_1$. Then $f(a) = x, f(b) = y, f(c) = z$ for some $a, b, c \in R$, it follows that $\eta$ is an interval-valued fuzzy weak bi-ideal of $\Gamma$-near-ring $R_1$

\[
\eta(x - y) = \eta(f(a) - f(b)) \\
= \eta(f(a - b)) \\
= (C_f^{-1}(\eta))(a - b) \\
\geq \min\{C_f^{-1}(\eta)(a), C_f^{-1}(\eta)(b)\} \\
= \min\{\eta(f(a)), \eta(f(b))\} \\
= \min\{\eta(x), \eta(y)\}
\]

\[
\omega(x - y) = \omega(f(a) - f(b)) \\
= \omega(f(a - b)) \\
= (C_f^{-1}(\omega))(a - b) \\
\leq \max\{C_f^{-1}(\omega)(a), C_f^{-1}(\omega)(b)\} \\
= \max\{\omega(f(a)), \omega(f(b))\} \\
= \max\{\omega(x), \omega(f(y))\}
\]

\[
\eta(x\alpha y\beta z) = \eta(f(a)\alpha f(b)\beta f(c)) \\
= \eta(f(aab\beta c)) \\
= (C_f^{-1}(\eta))(aab\beta c) \\
\geq \min\{C_f^{-1}(\eta)(a), C_f^{-1}(\eta)(b), C_f^{-1}(\eta)(c)\} \\
= \min\{\eta(f(a)), \eta(f(b)), \eta(f(c))\} \\
= \min\{\eta(x), \eta(y), \eta(z)\}
\]

\[
\omega(x\alpha y\beta z) = \omega(f(a)\alpha f(b)\beta f(c)) \\
= \omega(f(aab\beta c)) \\
= (C_f^{-1}(\omega))(aab\beta c) \\
\leq \max\{C_f^{-1}(\omega)(a), C_f^{-1}(\omega)(b), C_f^{-1}(\omega)(c)\} \\
= \max\{\omega(f(a)), \omega(f(b)), \omega(f(c))\} \\
= \max\{\omega(x), \omega(y), \omega(z)\}
\]

Hence, $A = \langle \eta, \omega \rangle$ is a cubic weak bi-ideal of $\Gamma$-near-ring $R_1$. \hfill \Box

**Theorem 4.6.** Let $f : R \to R_1$ be an onto $\Gamma$-near-ring homomorphism. If $A = \langle \eta, \omega \rangle$ is a cubic weak bi-ideal of $\Gamma$-near-ring $R$ then $C_f(A) = \langle C_f(\eta), C_f(\omega) \rangle$ is a cubic weak bi-ideal of $\Gamma$-near-ring $R_1$.

Proof. Let $A = \langle \eta, \omega \rangle$ is a cubic weak bi-ideal of $\Gamma$-near-ring $R$.

Since $C_f(\eta)(x') = \sup_{f(x) = x'} (\eta(x))$ for $x' \in R_1$ and $C_f(\omega)(x') = \inf_{f(x) = x'} (\omega(x))$ for $x' \in R_1$. 

So, \(C_f(A) = \langle C_f(\eta), C_f(\omega) \rangle\) is non-empty. Let \(x', y', z' \in R_1\). Then we have

\[
C_f(\eta)(x' - y') = \sup_{f(p) = x' - y'} \eta(p)
\]

\[
\geq \sup_{f(x) = x', f(y) = y'} \eta(x - y)
\]

\[
\geq \sup_{f(x) = x', f(y) = y'} \min \{\eta(x), \eta(y)\}
\]

\[
= \min \left\{ \sup_{f(x) = x'} \eta(x), \sup_{f(y) = y'} \eta(y) \right\}
\]

\[
= \min \{C_f(\eta)(x'), C_f(\eta)(y')\}
\]

\[
C_f(\omega)(x' - y') = \inf_{f(p) = x' - y'} \omega(p)
\]

\[
\leq \inf_{f(x) = x', f(y) = y'} \omega(x - y)
\]

\[
\leq \inf_{f(x) = x', f(y) = y'} \max \{\omega(x), \omega(y)\}
\]

\[
= \max \left\{ \inf_{f(x) = x'} \omega(x), \inf_{f(y) = y'} \omega(y) \right\}
\]

\[
= \max \{C_f(\omega)(x'), C_f(\omega)(y')\}
\]

\[
C_f(\eta)(x' \alpha y' \beta z') = \sup_{f(p) = x' \alpha y' \beta z'} \eta(p)
\]

\[
\geq \sup_{f(x) = x', f(y) = y', f(z) = z'} \eta(x\alpha y\beta z)
\]

\[
\geq \sup_{f(x) = x', f(y) = y', f(z) = z'} \min \{\eta(x), \eta(y), \eta(z)\}
\]

\[
= \min \left\{ \sup_{f(x) = x'} \eta(x), \sup_{f(y) = y'} \eta(y), \sup_{f(z) = z'} \eta(z) \right\}
\]

\[
= \min \{C_f(\eta)(x'), C_f(\eta)(y'), C_f(\eta)(z')\}
\]

\[
C_f(\omega)(x' \alpha y' \beta z') = \inf_{f(p) = x' \alpha y' \beta z'} \omega(p)
\]

\[
\leq \inf_{f(x) = x', f(y) = y', f(z) = z'} \omega(x\alpha y\beta z)
\]

\[
\leq \inf_{f(x) = x', f(y) = y', f(z) = z'} \max \{\omega(x), \omega(y), \omega(z)\}
\]

\[
= \max \left\{ \inf_{f(x) = x'} \omega(x), \inf_{f(y) = y'} \omega(y), \inf_{f(z) = z'} \omega(z) \right\}
\]

\[
= \max \{C_f(\omega)(x'), C_f(\omega)(y'), C_f(\omega)(z')\}
\]

Hence, \(C_f(A) = \langle C_f(\eta), C_f(\omega) \rangle\) is a cubic weak bi-ideal of \(\Gamma\)-near-ring \(R_1\). □

References


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