Global Hyperbolicity in Space-time Manifold

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Abstract

Global hyperbolicity is the most important condition on causal structure space-time, which is involved in problems as cosmic censorship, predictability etc. An open set *O* is said to be globally hyperbolic if, i) for every pair of points *x* and *y* in *O* the intersection of the future of *x* and the past of *y* has compact closure i.e., a space-time (M, g) is said to be globally hyperbolic if the sets $J^+(x) \cap J^-(y)$ are compact for all $x, y \in M$ (i.e., no naked singularity can exist in space-time topology), and ii) strong causality holds on *O* i.e., there are no closed or almost closed time like curves contained in *O*. Here $J^+(x)$ is causal future and $J^-(x)$ is the causal past of an event *x*. If a space-time is timelike or null geodesically incomplete but cannot be embedded in a larger space-time then we say that it has a singularity. An attempt is taken here to discuss global hyperbolicity and space-time singularity by introducing definitions, propositions and displaying diagrams appropriately.

Keywords: Cauchy surface, causality, global hyperbolicity, space-time manifold, space-time singularities.

1. Introduction

We consider a manifold *M* which is smooth i.e., *M* is differentiable as permitted by *M*. We assume that *M* is Hausdorff and paracompact.

Each generator of the boundary of the future has a past end point on the set one has to impose some global condition on the causal structure. Global hyperbolicity is the strongest and physically most important concept both in general and special relativity and also in relativistic cosmology. This notion was introduced by Jean Leray in 1953 (Leray 1953), and developed in the golden age of general relativity by A. Avez, B. Carter, Choquet-Bruhat, C. J. S. Clarke, Stephen W. Hawking, Robert P. Geroch, Roger Penrose, H. J. Seifert and others (Sánchez 2010). This is relevant to Einstein's theory of general relativity, and potentially to other metric gravitational theories. In 2003, Antonio N. Bernal and Miguel Sánchez showed that any globally hyperbolic manifold *M* has a smooth embedded 3-dimensional Cauchy surface, and furthermore that any two Cauchy surfaces for *M* are diffeomorphic (Bernal and Sánchez 2003, 2005).

Despite many advances on global hyperbolicity however, some questions which affected basic approaches to this concept, remained unsolved yet. For example, the so-called folk problems of smoothability, affected the differentiable and metric structure of any globally hyperbolic space-time M (Sachs and Wu 1977). The Geroch, Kronheimer and Penrose (GKP) causal boundary introduced a new ingredient for the causal structure of space-times, as well as a new viewpoint for global hyperbolicity (GKP 1972).

The existence of space-time singularities follows in the form of future or past incomplete non-spacelike geodesics in the space-time. Such a singularity would arise either in the cosmological scenarios, where it provides the origin of the universe or as the end state of the gravitational collapse of a massive star which has exhausted its nuclear fuel providing the pressure gradient against the inwards pull of gravity (Mohajan 2013c).

In the Schwarzschild metric and the Friedmann cosmological models solutions contained a space-time singularity where the curvature and density are infinite, and known all the physical laws would break down there. In the Schwarzschild solution such as a singularity was present at $r = 0$ which is the final fate of a massive star (Mohajan 2013b), whereas in the Friedmann model it was found at the epoch $t = 0$ (Big bang), which is the beginning of the universe, where the scale factor $S(t)$ also vanishes and all objects are crushed to zero volume due to infinite gravitational tidal force (Mohajan 2013a).

2. Some Related Definitions

In this section we introduce some definitions which we will use throughout this paper. The definitions are collected from the references of the reference list.

Manifold: A manifold is essentially a space which is locally similar to Euclidean space in that it can be covered by coordinate patches but which need not be Euclidean globally. Map $\phi : O \to O'$ where $O \subset R^n$ and $O' \subset R^m$ is said to be a class $C^r(r \geq 0)$ if the following conditions are satisfied. If we choose a point (event) p of coordinates $(x^1,...,x^n)$ on O and its image $\phi(p)$ of coordinates $(x'^1,...,x'^n)$ on O' then by C^r map we mean that the function ϕ is r-times differential and continuous. If a map is C^r for all $r \geq 0$ then we denote it by C^∞; also by C^0 map we mean that the map is continuous (Hawking and Ellis 1973, Mohajan 2015).

Hausdorff Space: A topological space M is a Hausdorff space if for pair of distinct points $p, q \in M$ there are disjoint open sets U_α and U_β in M such that $p \in U_\alpha$ and $q \in U_\beta$ (Joshi 1996).

Paracompact Space: An atlas $\{U_\alpha, \phi_\alpha\}$ is called locally finite if there is an open set containing every $p \in M$ which intersects only a finite number of the sets U_α. A manifold M is called a paracompact if for every atlas there is locally finite atlas $\{O_\beta, \psi_\beta\}$ with each O_β contained in some U_α. Let V^μ be a timelike vector, and then paracompactness of manifold M implies that there is a smooth positive definite Riemann metric $K_{\mu\nu}$ defined on M (Hawking and Ellis 1973).

Compact Set: A subset *A* of a topological space *M* is compact if every open cover of *A* is reducible to a finite cover (Hawking and Ellis 1973).

Tangent Space: A C^k-curve in *M* is a map from an interval of *R* in to *M* (figure 1). A vector $\left(\partial/\partial t\right)_{\lambda(t_0)}$ which is tangent to a C^1-curve $\lambda(t)$ at a point $\lambda(t_0)$ is an operator from the space of all smooth functions on *M* into *R* and is denoted by (Joshi 1996);

$$\left(\partial/\partial t\right)_{\lambda(t_0)}(f)=\left(\partial f/\partial t\right)_{\lambda(t_0)}=\underset{s\to 0}{Lim}\frac{f[\lambda(t+s)]-f[\lambda(t)]}{s}.$$

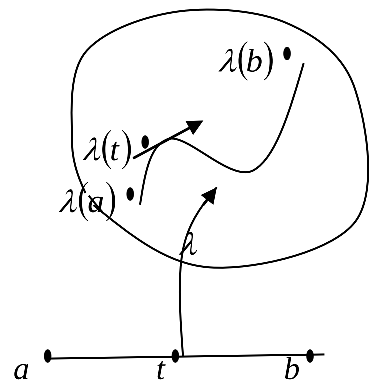

Figure 1: A curve in a differential manifold.

If $\{x^i\}$ are local coordinates in a neighborhood of $p=\lambda(t_0)$ then,

$$\left(\partial f/\partial t\right)_{\lambda(t_0)}=\frac{dx^i}{dt}.\frac{\partial f}{\partial x^i}\bigg|_{\lambda(t_0)}.$$

Thus every tangent vector at $p\in M$ can be expressed as a linear combination of the coordinates derivates, $\left(\partial/\partial x^1\right)_p,...,\left(\partial/\partial x^n\right)_p$. Thus the vectors $\left(\partial/\partial x^i\right)$ span the vector space T_p. Then the vector space structure is defined by $(\alpha X+\beta Y)f=\alpha(Xf)+\beta(Yf)$. The vector space T_p is also called the tangent space at the point *p*.

A metric is defined as;

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu \qquad (1)$$

where $g_{\mu\nu}$ is an indefinite metric in the sense that the magnitude of non-zero vector could be either positive, negative or zero. Then any vector $X\in T_p$ is called timelike, null, spacelike or non-spacelike respectively if;

$$g(X,X)<0,\ g(X,X)=0,\ g(X,X)>0,\ g(X,X)\le 0. \qquad (2)$$

Orientation: Let *B* be the set of all ordered basis $\{e_i\}$ for T_p, the tangent space at point *p*. If $\{e_i\}$ and $\{e_j\}$ are in *B*, then we have $e_{j'} = a^i_{j'} e_i$. If we denote the matrix (a_{ij}) then $\det(a) \neq 0$. An *n*-dimensional manifold *M* is called orientable if *M* admits an atlas $\{U_i, \varphi_i\}$ such that whenever $U_i \cap U_j \neq \varphi$ then the Jacibian, $J = \det\left(\frac{\partial x^i}{\partial x^j}\right) > 0$, where $\{x^i\}$ and $\{x^j\}$ are local coordinates in U_i and U_j respectively. The Möbious strip is a non-orientable manifold. A vector defined at a point in Möbious strip with a positive orientation comes back with a reversed orientation in negative direction when it traverses along the strip to come back to the same point (Mohajan 2015).

Space-time Manifold: General Relativity models the physical universe as a 4-dimensional C^∞ Hausdorff differentiable space-time manifold *M* with a Lorentzian metric *g* of signature $(-,+,+,+)$ which is topologically connected, paracompact and space-time orientable. These properties are suitable when we consider for local physics. As soon as we investigate global features then we face various pathological difficulties such as, the violation of time orientation, possible non-Hausdorff or non-papacompactness, disconnected components of space-time etc. Such pathologies are to be ruled out by means of reasonable topological assumptions only. However, we like to ensure that the space-time is causally well-behaved. We will consider the space-time Manifold (M, g) which has no boundary. By the word ‘boundary’ we mean the ‘edge’ of the universe which is not detected by any astronomical observations. It is common to have manifolds without boundary; for example, for two-spheres S^2 in R^3 no point in S^2 is a boundary point in the induced topology on the same implied by the natural topology on R^3. All the neighborhoods of any $p \in S^2$ will be contained within S^2 in this induced topology. We shall assume *M* to be connected i.e., one cannot have $M = X \cup Y$, where *X* and *Y* are two open sets such that $X \cap Y \neq \varphi$. This is because disconnected components of the universe cannot interact by means of any signal and the observations are confined to the connected component wherein the observer is situated. It is not known if *M* is simply connected or multiply connected. Manifold *M* is assumed to be Hausdorff, which ensures the uniqueness of limits of convergent sequences and incorporates our intuitive notion of distinct space-time events (Joshi 1996).

Hypersurface: In the Minkowski space-time $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, the surface $t = 0$ is a three-dimensional surface with the time direction always normal to it. Any other surface $t = \text{constant}$ is also a spacelike surface in this sense. Let *S* be an $(n-1)$-dimensional manifold. If there exists a C^∞ map $\phi : S \to M$ which is locally one-one i.e., if there is a neighborhood *N* for every $p \in S$ such that ϕ restricted to *N* is one-one, and ϕ^{-1} is a C^∞ as defined on $\phi(N)$, then $\phi(S)$ is called an embedded sub-manifold of *M*. A hypersurface *S* of any *n*-dimensional manifold

M is defined as an $(n-1)$-dimensional embedded sub-manifold of M. Let V_p be the $(n-1)$-dimensional subspace of T_p of the vectors tangent to S at any $p \in S$ from which follows that there exists a unique vector $n^a \in T_p$ and is orthogonal to all the vectors in V_p. n^a is called the normal to S at p. If the magnitude of n^a is either positive or negative at all points of S without changing the sign, then n^a could be normalized so that $g_{ab}n^a n^b = \pm 1$. If $g_{ab}n^a n^b = -1$ then the normal vector is timelike everywhere and S is called a spacelike hypersurface. If the normal is spacelike everywhere on S with a positive magnitude, S is called a timelike hypersurface. Finally, S is null hypersurface if the normal n^a is null at S (Mohajan 2015).

3. Causality and Chronology in Space-time (M, g)

In Lorentzian geometry causality plays an important role, as it displays relativistic interpretation of space-time for both special and general relativity. Causality also appears as a fruitful interplay between relativistic motivations and geometric developments. Causal space-time is established at the end of the 1970s, after the works of Carter, Geroch, Hawking, Kronheimer, Penrose, Sachs, Seifert, Wu and others (Hawking and Sachs 1974). No material particle can travel faster than the velocity of light. Hence, causality fixes the boundary of the space-time topology.

We assume that the timelike curves to be smooth; with future-directed tangent vectors everywhere strictly timelike, including its end-points. A causal curve is a curve in space-time which is nowhere spacelike. A causal curve is continuous but not necessarily everywhere smooth; its tangent vectors are either timelike or null. A causal curve will required end-points if it can be extended as a causal curve either into the past or the future. If a causal curve can be extended indefinitely and continuously into the past then it is called past-inextensible. The future-inextensible curve is defined similarly. If a causal curve is both past and future-inextensible then it is called simply inextensible (Hawking and Penrose 1970). An event x chronologically precedes another event y, denoted by $x << y$, if there is a smooth future directed timelike curve from x to y. If such a curve is non-spacelike then x causally precedes y i.e., $x < y$. The chronological future $I^+(x)$ be the set of all points of the space-time M that can be reached from x by future directed timelike curves. We can think of $I^+(x)$ as the set of all events that can be influenced by what happens at x. Now $I^+(x)$ and $I^-(x)$ of a point x are defined as (figure 2).

$$I^+(x) = \{\, y \in M \,/\, x << y \}, \text{ and}$$

$$I^-(x) = \{\, y \in M \,/\, y << x \}.$$

One can think of $I^+(x)$ as the set of all events that can be influenced by what happens at x. The causal future (past) of x can be defined as;

$$J^+(x) = \{\, y \in M \,/\, x < y \},$$

$$J^{-}(x)=\{y\in M / y<x\}.$$

Also $x<<y$ and $y<z$ or $x<y$ and $y<<z$ implies $x<<z$. Hence, the closer and boundary of $I^{+}(x)$ and past $I^{-}(x)$ of a point *x* are defined respectively as (Penrose 1972);

$\overline{I^{+}(x)}=\overline{J^{+}(x)}$ and $\dot{I}^{+}(x)=\dot{J}^{+}(y)$, where $\dot{I}$ is a topological boundary and $\bar{I}$ is the closure of *I*.

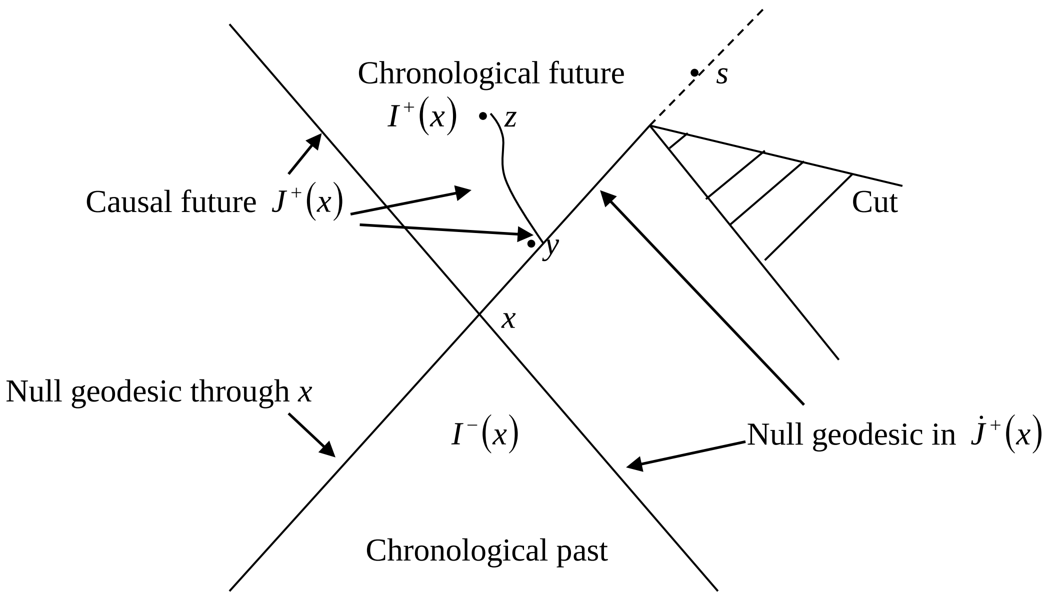

Figure 2: Removal of a closed set from the space-time gives a causal future $J^{+}(x)$ which is not closed. Events *x* and *s* are not causally connected.

Similarly the chronological (causal) future of any set $S\subset M$ is defined as;

$$I^{+}(S)=\bigcup_{x\in S} I^{+}(x), \text{ and}$$

$$J^{+}(S)=\bigcup_{x\in S} J^{+}(x).$$

The definitions of past subsets of space-time are dual.

The boundary of the future is null apart from at *S* itself. If *x* is in the boundary of the future but is not in the closure of *S* there is a past directed null geodesic segment through *x* lying in the boundary. Hence the boundary of the future of *S* is generated by null geodesics that have a future end point in the boundary and pass into the interior of the future if they intersect another generator and the null geodesic generators can have past end points only on *S* (Hawking 1994).

Proposition 1 (Penrose 1972)**:** The chronological future $I^{+}(x)$ and chronological past $I^{-}(x)$ are open sets.

Proof: The chronological future $I^{+}(x)$ does not contain all the future points of an event *x*. It contains only interior points of causal future $J^{+}(x)$, i.e., it does not contain null geodesics of space-time. Hence $I^{+}(x)$ is an open set. Similarly, past $I^{-}(x)$ is an open set. ■

Proposition 2: The causal future $J^{+}(x)$ and causal past $J^{-}(x)$ are neither closed nor open.

Proof: The causal future $J^{+}(x)$ is closed, since it contains all the points of timelike and null geodesics. But from the figure 2 we have seen that if a point *s* of $J^{+}(x)$ is deleted then $J^{+}(x)$ is no more closed. On the other hand $J^{+}(x)$ is not open since it not only contains interior of the space-time but also the boundary i.e., it also contains points of null geodesics. Hence, the causal future $J^{+}(x)$ is neither closed nor open. Similarly the causal past $J^{-}(x)$ is neither closed nor open. ■

4. The Globally Hyperbolic Space-Time

Now we provide some definitions before the discussion of the globally hyperbolic space-time.

Causally Convex Set

Let *S* and *T* be open subsets of a space-time (M, g), with $T \subset S$ then *T* is called causally convex in *S* if any causal curve contained in *S* with endpoints in *T* is entirely contained in *T*. In particular, when this holds for $S = M$, *T* is called *causally convex*. Again if *T* is causally convex in *S* and *U* is an open set such that $T \subset U \subset S$, then *T* is causally convex in *U* (Minguzzi and Sánchez 2008).

Future Set and Past Set

An open subset *F* is a future set if $I^{+}(F) = F$. The past set *P* is defined by $I^{-}(P) = P$. The boundary of a future set *F* is made of all events *x* such that $I^{-}(x) \subset F$ but $x \notin F$. If $x \in \dot{F}$ then of course $x \notin F$, since *F* is an open set.

Achronal Set

A set *S* in *M* is said to be achronal if no two points $x, y \in S$ may be joined by a piecewise timelike curve i.e., there do not exist $x, y \in S$ such that $y \in I^{+}(x)$. Let *F* be a future set, then the boundary of *F* is a closed, achronal C^{0}-manifold that is a 3-dimensional embedded hypersurface.

Domain of Dependence of a Set

The future domain of dependence (the future Cauchy development) of a spacelike three-surace *S*, denoted by $D^{+}(S)$, is defined as the set of all points $x \in M$ such that every past-inextendible non-spacelike curve from *x* intersects *S*, i.e., $D^{+}(S)$ = {*x*: every past-inextensible timelike curve through *x* meets *S*}. It is clear that $S \subset D^{+}(S) \subset J^{+}(S)$ and *S* being achronal, $D^{+}(S) \cap I^{-}(S) = \varphi$. The past domain of dependence $D^{-}(S)$ is defined similarly. The full domain of dependence for *S* is defined as; $D(S) = D^{+}(S) \cup D^{-}(S)$ (Joshi 1993).

Cauchy Surface

Let *S* be a closed achronal set. The edge of *S* is defined as a set of points $x \in S$ such that every neighborhood of *x* contains $y \in I^{+}(x)$ and $z \in I^{-}(x)$ with a timelike curve from *z* to *y* which does not meet *S*. A partial Cauchy surface *S* is defined as an acausal set without an edge. So that no non-spacelike curve intersects *S* more than once and *S* is a spacelike hypersurface.

A partially Cauchy surface is called a Cauchy surface *S* or a global Cauchy surface if $D(S) = M$ i.e., if a set *S* is closed, achronal, and its domain of dependence is all of the space-time, $D(S) = M$. In another way, if $D(S) = M$ i.e., if every inextensible non-spacelike curve in intersect *S*, then *S* is said to be a Cauchy surface (figure 3). For a Cauchy surface *S*, $edge(S) = \varphi$. The Cauchy development is the region of spacetime that can be predicted from data on S. Here *S* must be an embedded topological hypersurface and must be also crossed by any inextensible causal curve γ (Hawking 1966a,b). The existence of a Cauchy hypersurface *S* implies that *M* is homeomorphic to $t \times S$, and all Cauchy hypersurfaces are homeomorphic.

Every non-spacelike curve in *M* meets *S* once and exactly once if *S* is a Cauchy surface. The relationship between the global hyperbolicity of *M* and the notion of Cauchy surface is shown in figure 3 (Hawking and Ellis 1973):

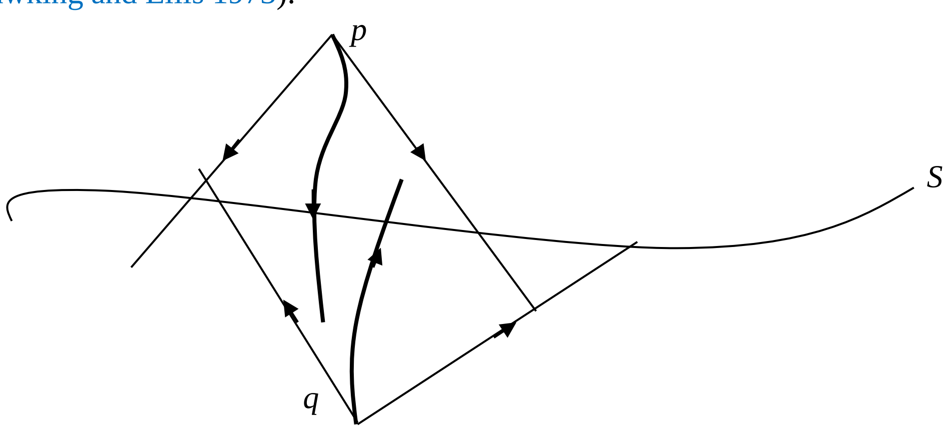

Figure 3: The spacelike hypersurface *S* is a Cauchy surface in the sense that for any *p* in future of *S*, all past non-spacelike curves from *p* intersect *S*. The same holds for all future directed curves from any point *q* in past of *S*.

Time function is a continuous function $t: M \to R$ which increases strictly on any future-directed causal curve. If the levels $t = \text{constant}$ are Cauchy hypersurfaces, then *t* is a Cauchy time function. The space-time manifold has a Cauchy surface *S*.

A space-time (M, g) is said to be metrically complete if every Cauchy sequence with respect to the metric converges to a point in *M*.

4.1 Globally Hyperbolicity

In mathematical physics, global hyperbolicity is a certain condition on the causal structure of a space-time manifold. If *M* is a smooth connected Lorentzian manifold with boundary, we say it is globally hyperbolic if its interior is globally hyperbolic. Penrose has called globally hyperbolic space-times "the physically reasonable space-times" (Wald 1984). A space-time (M, g) which admits a Cauchy surface is called globally hyperbolic.

A space-time (M, g) which admits a Cauchy surface is called globally hyperbolic. An open set *O* is said to be globally hyperbolic if, i) for every pair of points *x* and *y* in *O* the intersection of the future of *x* and the past of *y* has compact closure i.e., if a space-time (M, g) is said to be globally hyperbolic if the sets $J^+(x) \cap J^-(y)$ are compact for all $x, y \in M$ (i.e., no naked singularity can exist in space-time topology). In other words, it is a bounded diamond shaped region (diamond-compact) and ii) strong causality holds on *O* i.e., there are no closed or almost closed time like curves contained in *O* (figure 3). Then it also satisfies that $J^+(x)$ and $J^-(y)$ are closed $\forall x, y \in M$. More precisely, consider two events *x*, *y* of the space-time (M, g), and let $C(x, y)$ be the set of all the continuous curves which are future-directed and causal and connect *x* with *y* (Hawking and Ellis 1973).

Proposition 3: If a globally hyperbolic space-time (M, g) is such that sets $J^+(x) \cap J^-(y)$ are compact for all $x, y \in M$, then $J^+(x)$ and $J^-(y)$ are closed $\forall x, y \in M$.

Proof: Suppose $J^+(x)$ is not a closed set in a globally hyperbolic space-time (M, g). Now let an event $p \in \bar{J}^+(x) - J^+(x)$ and an another event $q \in I^+(x)$, where $I^+(x)$ is an open set. Now a set for a sequence of points $\{p_n\} \to p$. On the other hand we have $p_n << y$ for some finite *n*. We know $\{p_n\}$ is a subset of compact set $J^+(x) \cap J^-(y)$ and converges to event *p*. But *p* does not meet $J^-(y)$ (figure 4). We have arrived in a contradiction. Hence $J^+(x)$ is a closed set. Similarly, $J^-(y)$ is closed a set. ■

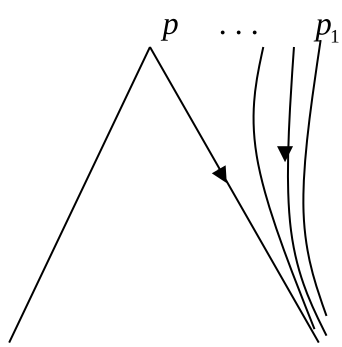

Figure 4: The sequence of points $\{p_n\}$ does not meet $J^-(y)$.

If S_1 and S_2 are any two compact subsets, $J^+(S_1) \cap J^-(S_2)$ must be compact. Geroch (1970) proved that global hyperbolicity is equivalent to the existence of a topological Cauchy surface and that the space-time manifold is homeomorphic to the product manifold $M \times \Sigma$, where Σ is the topological Cauchy surface. A globally hyperbolic space-time must be causally simple. In globally hyperbolic space-time strong causality must exists. Globally hyperbolicity is strong on M which uniquely fixes the overall topology of the space-time.

Minkowski space-time, de Sitter space-time and the exterior Schwarzschild solution, Friedmann, Robertson-Walker (FRW) cosmological solutions and the steady state models are all globally hyperbolic. The Kerr solution is not globally hyperbolic, since it represents rotating model i.e., not static model. On the other hand anti de Sitter space-time and the Godel universe are not globally hyperbolic. The global hyperbolicity of M is closely related to the future or past development of initial data from a given spacelike hypersurface (Joshi 1996).

The physical significance of global hyperbolicity comes from the fact that it implies that there is a family of Cauchy surfaces $\Sigma(t)$ for globally hyperbolic open set O. A Cauchy surface for O is a spacelike or null surface that intersects every timelike curve in O once and only once. Let x and y be two points of O that can be joined by a timelike or null curve, then there is a timelike or null geodesic between x and y which maximizes the length of timelike or null curves from x to y (Hawking 1994).

4.2 Cauchy Horizons of a Set

Let S be a partial Cauchy surface. Then $N = D^+(S) \cup D^-(S) \neq M$ and N must be a proper subset of M. The boundary of N in M can be divided into two portions. Now suppose that the future Cauchy development was compact. This would imply that the Cauchy development would have a future boundary called the Cauchy horizon, $H^+(S)$. Since the Cauchy development is assumed to be compact, the Cauchy horizon will also be compact. The $H^+(S)$ and $H^-(S)$ which are respectively called the future and past Cauchy horizons of S. We can write (Hawking and Penrose 1970);

$$H^+(S) = \{ x / x \in D^+(S), I^+(x) \cap D^+(S) = \varphi \}$$

$$= D^+(S) - I^-[D^+(S)].$$

$H^-(S)$ is defined similarly. $H^+(S)$ is an achronal closed set. Also we can write, $I^+[H^+(S)] = I^+[S] - D^+(S)$.

The Cauchy horizon will be generated by null geodesic segments without past end points. Even though *M* may not be globally hyperbolic and *S* is not a Cauchy surface, the region $Int(D^+(S))$ or $Int(D^-(S))$ is globally hyperbolic in its own right and the surface *S* serves as a Cauchy surface for the manifold $Int(N)$. Thus $H^+(S)$ or $H^-(S)$ represents the failure of *S* to be global Cauchy surface for *M* (figure 5).

If every geodesic can be extended to arbitrary values of its affine parameter then it is geodesically complete. If a timelike or causal curve can be extended indefinitely and continuously into the past (future) then it is called past-inextensible (future-inextensible).

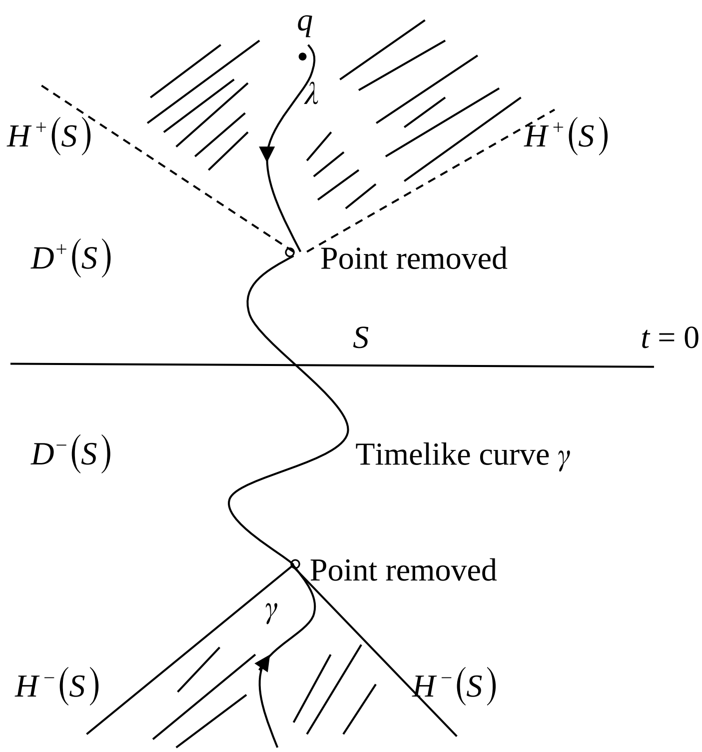

Figure 5: The space-time obtained by removing a point from the Minkowski space-time is not globally hyperbolic. The point *q* does not meet *S* in the past. The event $p \in D^+(S)$. The Cauchy horizon is the boundary of the shaded region which consists of points not in $D^+(S)$.

In globally hyperbolic space-times, there is a finite upper bound on the proper time lengths of non-spacelike curves two chronologically related events. Of course there is no lower limit of length for such curves except zero, because the chronologically related events can always be joined using broken null curves which could give an arbitrary small length curve between them.

If S is Cauchy surface in globally hyperbolic space-time M, then for any point p in the future of S, there is a past directed timelike geodesic from p orthogonal to S which maximizes the lengths of all non-spacelike curves from p to S (figure 6).

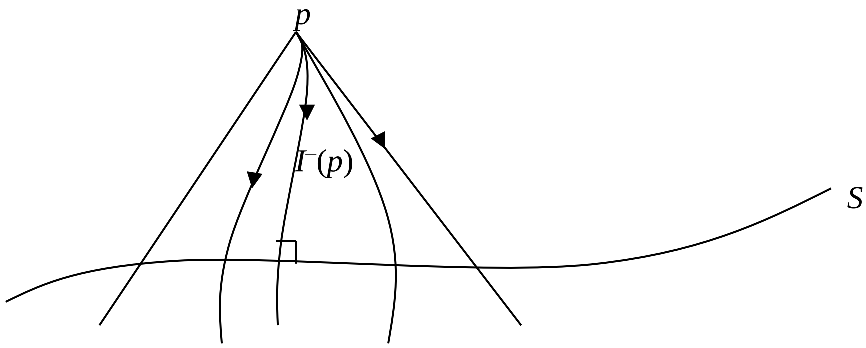

Figure 6: The spacelike hupersurface S is a Cauchy surface in the sense that for any p in future of S, all past directed non-spacelike curves from p intersect S.

An important property of globally hyperbolic space-time that is relevant for the singularity theorems is the existence of maximum length non-spacelike geodesics between pair of causally related events. In a complete Riemannian manifold with a positive definite metric any two points can be joined by a geodesic of minimum length and in fact such a geodesic need not be unique (Joshi 1996). (In a sphere paths of great circles are geodesics. Opposite poles can be joined by an infinite numbers of geodesics.)

5. Space-time Singularities

If a space-time is timelike or null geodesically incomplete but cannot be embedded in a larger space-time then we say that it has a singularity. Einstein's field equation can be written as (Mohajan 2013c);

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}. \quad (3)$$

Here $G = 6.673 \times 10^{-11}$ is the gravitational constant, $c = 10^8\, m/s$ is the velocity of light, $T_{\mu\nu}$ is the energy momentum tensor and $g_{\mu\nu}$ is an indefinite metric defined above. From (3) we can write Einstein's empty space equation as;

$$R_{\mu\nu} = 0. \quad (4)$$

By (4) the Schwarzschild solution in (t, r, θ, ϕ) coordinates is given by (Mohajan 2013b);

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) - c^2\left(1 - \frac{2m}{r}\right) dt^2 \quad (5)$$

which is extensively used for experimental verification of general relativity.

From the Schwarzschild metric (5), there are singularities at $r = 0$ and $r = 2m$, because one of the $g^{\mu\nu}$ or $g_{\mu\nu}$ is not continuously defined. Here $r = 0$ is a real singularity in the sense that along any non-spacelike trajectory falling into the singularity as $r \to \infty$ the Kretschman scalar $\alpha = R^{\mu\nu\gamma\sigma} R_{\mu\nu\gamma\sigma}$ tends to infinity and $r = 2m$ is a coordinate singularity (Kruskal 1960 and Szekeres 1960).

In (t, r, θ, ϕ) coordinates the Robertson-Walker (R-W) line element is given by;

$$ds^2 = -dt^2 + S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \qquad (6)$$

where k is a constant which denotes the spatial curvature of the three-space and could be normalized to the values +1, 0, –1. Again in R-W models the Einstein equation (3) imply that $\rho + 3p > 0$ at all times, where ρ is the total density and p is the pressure, there is a singularity at $t = 0$, since $S^2(t) \to 0$ when $t \to 0$ in the sense that curvature scalar $\hat{R} = R^{\mu\nu} R_{\mu\nu}$ bends to infinity. Here we consider the time $t = 0$ is the beginning of the universe. Thus there is an essential curvature singularity at $t = 0$ which cannot be transformed away by any coordinate transformation (Mohajan 2013a). The existences of real singularities where the curvature scalars and densities diverge imply that all the physical laws break down. Let us consider the metric;

$$ds^2 = -\frac{1}{t^2} dt^2 + dx^2 + dy^2 + dz^2 \qquad (7)$$

which is singular on the plane $t = 0$. If any observer starting in the region $t > 0$ tries to reach the surface $t = 0$ by traveling along timelike geodesics, he will not reach at $t = 0$ in any finite time, since the surface is infinitely far into the future. If we put $t' = \ln(-t)$ in $t < 0$ then (7) becomes;

$$ds^2 = -dt'^2 + dx^2 + dy^2 + dz^2 \qquad (8)$$

with $-\infty < t' < \infty$ which is Minkowski metric and there is no singularity at all, which is a removable singularity like Schwarzschild singularity at $r = 2m$. Let us consider a non-spacelike geodesic which reaches the singularity in a proper finite time. Such a geodesic will have not any end point in the regular part of the space-time. A timelike geodesic which, when maximally extended, has no end point in the regular space-time and which has finite proper length, is called timelike geodesically incomplete (Clarke 1986).

A point $p \in M$ is said to be a singular point on a geodesic γ of the congruence if expansion θ is infinity on γ at *p*. A space-time is singular if it contains an incomplete curve $\gamma : [0, a) \to M$

such that there is no extension $\theta : M \rightarrow M'$ for which $\theta \circ \gamma$ is extensible. Hence the region $r > 2m$ in the Schwarzschild solution (5) is not singular, merely incomplete. Singular points of congruences are points where infinitesimally neighboring geodesics meet (Mohajan 2013c).

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